Non-intersecting random walks in the neighborhood of a symmetric tacnode

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Abstract

Consider a continuous time random walk in $\mathbb{Z}$ with independent and exponentially distributed jumps ± 1. The model in this paper consists in an infinite number of such random walks starting from the complement of $\{-m, -m+1, ..., m-1, m\}$ at time $-t$, returning to the same starting positions at time t, and *conditioned not to intersect*. This yields a determinantal process, whose gap probabilities are given by the Fredholm determinant of a kernel. Thus this model consists of two groups of random walks, which are contained into two ellipses which, with the choice $m \simeq 2t$ to leading order, just touch: so we have a *tacnode*. We determine the new limit extended kernel under the scaling $m = \lfloor 2t + \sigma t^{1/3} \rfloor$, where parameter σ controls the strength of interaction between the two groups of random walkers.

1 Introduction

In the past decade, systems of vicious random walks and non-intersecting Brownian motions have been investigated, and quantities such as the correlation functions [38], the one-point distribution functions and limit processes under appropriate scaling limits have been studied. Non-intersecting Brownian motions arise in the study of random matrices [32, 33, 37], and

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space (and/or) time discrete versions in random tiling and growth models [23,24,26–28,39,41,42]. Most of these works use the mathematical framework shared by Brownian motions starting from a point, and either ending at the same point after a given time or the boundary condition is free (with possible extra boundary conditions like staying positive [36, 50]).

Consider N non-intersecting Brownian bridges $x_i(\tau)$ on $\mathbb{R}$, leaving from 0 at time $\tau = -2N$ and forced to 0 at time $\tau = 2N$. For large N, the mean density of Brownian paths has support, for each $-2N < \tau < 2N$, on the interval $(-\sqrt{4N^2 - \tau^2}, \sqrt{4N^2 - \tau^2})$. This means that on the macroscopic scale, where space and time unit are set equal to N, one sees a circle. Near its boundary, the density of Brownian paths is of order $N^{-1/3}$, thus to see something non-trivial one needs to look in a space window of size $N^{1/3}$ and, by Brownian scalings, a time window of size $N^{2/3}$. We call this the "*Airy microscope*", since it holds

$$\lim_{N\to\infty} \mathbb{P}\left(\text{all } N^{-1/3}\left(x_i\left(2sN^{2/3}\right) - 2N\right) \in E^c - s^2\right) = \mathbb{P}\left(\mathcal{A}_2(s) \cap E = \emptyset\right), \tag{1.1}$$

where $\mathcal{A}_2$ is the so-called **Airy$_2$ process**. It has a *universal* character and was discovered in the context of the so-called multilayer PNG model [41]. The scaling (1.1) is equivalent to the customary $N^{-1/6}$-GUE-edge rescaling along the circle for non-intersecting Brownian motions leaving from the origin at time $t = 0$ and returning to the origin at time $t = 1$; this is done by an appropriate change of the variance of the Brownian motions.

In the context of growth models, generalizations have been introduced with external sources [11, 25]. Its analogue in terms of Brownian motions is to require that a finite number of Brownian motions end up at some point αN. Then under the scaling in (1.1), the limit process is a transition process from Airy$_2$ to Brownian Motion. For extensions to more general sources, see [9, 18], while for the case that the top r Brownian motion end up at $2N$, see [3] and [4].

A further known situation occurs when a fraction pN of the N non-intersecting Brownian motions (leaving from the origin at time $t = 0$) end at time $t = 1$ at position aN and another fraction $(1 - p)N$ at bN, with $a < b$. When $N \to \infty$, the mean density of Brownian particles has its support on one interval in the beginning and on two intervals near the end. Thus a bifurcation appears for some intermediate time τ_0, where one interval splits into two intervals, creating a "heart-like" shape with a cusp at the origin. Near this cusp appears a new *universal* process, upon looking through the "*Pearcey microscope*", where the space window is $N^{1/4}$ and the time window is $N^{1/2}$. The new process is is called the **Pearcey process** [47] and is independent of the values of a, b and p; see [6]. Once the bifurcation has

taken place, the Brownian motions will eventually fluctuate like the Airy_2 process near the edge, with a transition from the Pearcey to the Airy_2 process [2]. The Pearcey process has been obtained as the limit of discrete models, see [14, 15, 40].

The motivation of our work was to understand what happens when half of the non-intersecting Brownian motions start and end at a point, while the second half start and end at another point. When the two starting points are sufficiently far apart from each other, the mean density of particles will be confined to two separate circles, with Airy_2 processes appearing near the boundary, as described above. When the two starting points move away from each other at an appropriate rate proportional to N, the two circles will just touch, creating a tacnode. A new *critical* process appears by looking at the two sets of non-intersecting Brownian motions, which experience a brief meeting in the neighborhood of the tacnode, but looked at with the *Airy scaling*; we call it the **tacnode process**. Pictorially it can be thought of as two Airy_2 processes touching.

In this paper we obtain an explicit formula for the kernel governing this tacnode process starting for non-intersecting continuous-time random walks, rather than non-intersecting Brownian motions. The same result is expected to hold for the Brownian motion case, since under the scaling the discrete nature of the random walks is lost and the random walks becomes Brownian paths. Our main result is the limiting kernel at the tacnode under appropriate scaling limit, stated in Theorem 2.2. Before taking the limit, the kernel is given by Theorem 2.1. The model is to let two groups of non-intersecting random walks with jumps ± 1, rate 1 and $2m+1$ integers apart evolve during a total time or order m, with space-time rescaled *à la Airy*, namely $x \sim \xi m^{1/3}$ and $\tau \sim s m^{2/3}$ as suggested by formula (1.1).

There is an important difference with respect to the previous two cases: here we have a one-parameter family of processes, which is obtained by modulating the end points distance between the two sets of Brownian motions over distance of order $N^{1/3}$. For the Pearcey processes (and the Airy_2 process), geometric changes of this type only have the effect to modify the position (and orientation) of the cusp, but the underlying Pearcey process remains unchanged. In the literature there is another known situation with a process in a tacnode-like geometry [14], which however differs from the present one.

This problem can be approached using multiple orthogonal polynomials [20] and a Riemann-Hilbert to this problem is analyzed in a paper [21] (which meanwhile appeared in the arXiv). In a forthcoming paper [30] Johansson uses a different approach leading to a different kernel for the Brownian motion problem, which is expected to be the same as the kernel in our paper. Adler, Johansson and van Moerbeke have then considered two

partially overlapping Aztec diamonds and found the same kernel [5].

Outline

In Section 2 we define the model and state the two main results. In Section 3, Theorem 3.1, we derive the finite time result for $\tau = 0$, which is reshaped in Section 4 as a preparation to carrying out the large time limit. Before actually doing this, we indicate in Section 5 how to introduce the time, leading to the finite multi-time kernel in Theorem 5.4, an extension of the kernel appearing in Proposition 4.1. In Section 6, we take the limit of the multi-time kernel, leading to the proof of the first formula of Theorem 2.2. In Section 7, we sketch the proof of the double integral representation of the kernel, the second formula of Theorem 2.2, using the steep descent analysis.

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2 Model and results

Consider a continuous time random walk in $\mathbb{Z}$ with jumps ± 1, occuring independently with rate 1, i.e., the waiting times of the up- and down-jumps are independent and exponentially distributed with mean 1. The transition probability $p_t(x, y)$ of going from x to y during a time interval of length t is given by

$$p_t(x, y) = e^{-2t} I_{|x-y|}(2t), \tag{2.1}$$

where I_n is the modified Bessel function of degree n (see [1]).

Consider now an infinite number of continuous time random walks starting from $\{\ldots, -m-2, -m-1\} \cup \{m+1, m+2, \ldots\}$ at time $\tau = -t$, returning to the starting positions at time $\tau = t$, and *conditioned not to intersect*, see Figure 1. Denote $\tilde{x}_k(\tau)$ the position of the walk that starts and ends at position k. Then, the point process $\tilde{\eta}$ on $\mathbb{Z}$ (described by the little white circles

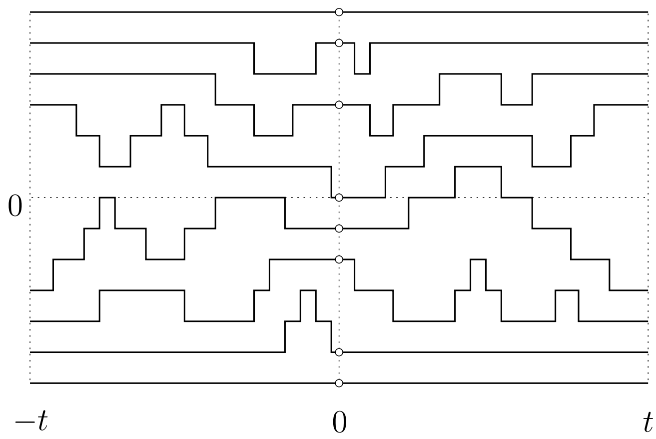

Figure 1: The lines are the non-intersecting walks $\tilde{\mathbf{x}}$. The white dots are the support of the point process $\tilde{\eta}$.

in Figure 1) defined by

$$\tilde{\eta}(x) = \sum_{k\in\mathbb{Z}\setminus\{-m,\dots,m\}} \delta_{x,\tilde{x}_k(0)}, \tag{2.2}$$

with δ the Kronecker-delta, is determinantal, i.e., there exists a kernel $\widetilde{\mathbb{K}}_m$ such that the k-point correlation function $\rho^{(k)}$ is given by $\rho^{(k)}(y_1,\dots,y_k) = \det(\widetilde{\mathbb{K}}_m(y_i,y_j))_{1\le i,j\le k}$. One of the interesting quantities is the *gap probability of a set* E, which is given by $\mathbb{P}(\tilde{\eta}(\mathbb{1}_E)=0)$, i.e., the probability that *none of the random walks are in* E at time $\tau=0$. For a determinantal point process the gap probability is given by the Fredholm determinant of the associated kernel $\widetilde{\mathbb{K}}_m$ projected onto E. For more informations on determinantal point processes, see [12, 29, 35, 44, 45].

The determinantal structure still holds if we consider the point process on a set of time-slices instead of a single time $\tau=0$. This means that given times $\tau_1<\tau_2<\ldots<\tau_p$ in the interval $(-t,t)$, the point process on $\{\tau_1,\dots,\tau_p\}\times\mathbb{Z}$ defined by

$$\tilde{\eta}(\tau,x) = \sum_{r=1}^{p} \sum_{k\in\mathbb{Z}\setminus\{-m,\dots,m\}} \delta_{(\tau,x),(\tau_r,\tilde{x}_k(\tau_r))}, \tag{2.3}$$

is determinantal. That is, the space-time correlation functions are given by the determinant of an extended kernel, which we denote by $\widetilde{\mathbb{K}}_m^{\rm ext}(t_1,x_1;t_2,x_2)$, where $t_i\in\{\tau_1,\dots,\tau_p\}$ and $x_i\in\mathbb{Z}$.

It is more convenient to first study the *dual* or complementary process $\mathbf{x}(\tau)$. The *dual proceeds along the gaps of* $\tilde{\mathbf{x}}(\tau)$. In this instance, the dual $\mathbf{x}(\tau)$

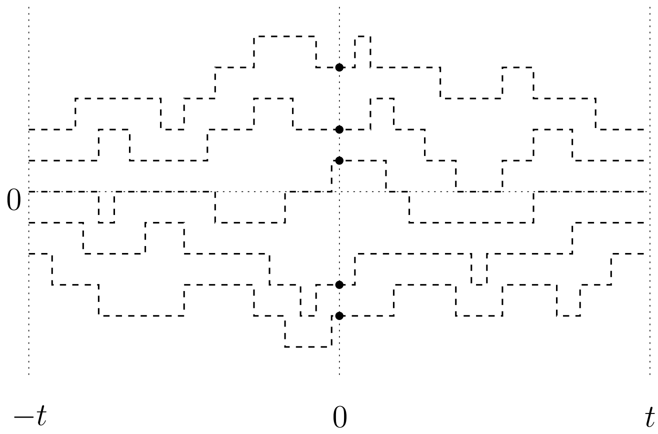

Figure 2: The solid lines are the non-intersecting walks $\mathbf{x}$, the dual process of $\tilde{\mathbf{x}}$ of Figure 1. The black dots are the support of the point process η.

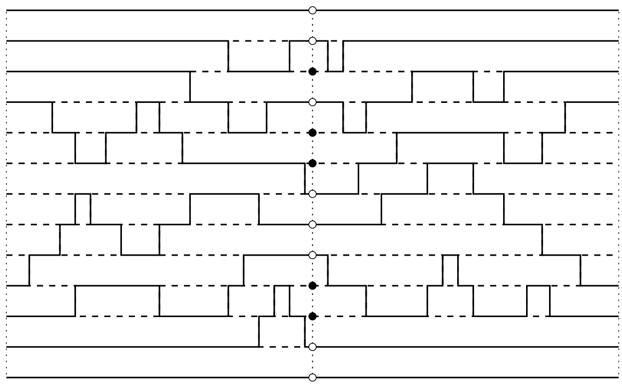

Figure 3: Superposition of Figure 1 and Figure 2

of $\tilde{\mathbf{x}}(\tau)$ is described by $n = 2m+1$ ($m \in \mathbb{N}$) *non-intersecting* continuous-time random walks, starting from $-m, -m+1, \ldots, m$ at time $\tau = -t$, returning to the starting positions at time $\tau = t$; see Figure 2, and Figure 3 for the superposition of the trajectories of $\mathbf{x}(\tau)$ and $\tilde{\mathbf{x}}(\tau)$.

In particular, the dual process $\mathbf{x}(\tau)$ at $\tau = 0$ is given by the the little black circles in Figure 2. The probability measure at time $\tau = 0$ is obtained by the Karlin-McGregor formula [31], and thus it is a determinantal process for a kernel $\mathbb{K}_m$. Finally, the complementation principle by Borodin, Olshanski, and Okounkov (see Appendix of [17]) tells us that, if the kernel $\mathbb{K}_m$ governs the process $\mathbf{x}(\tau)$, then the kernel $\widetilde{\mathbb{K}}_m = \mathbb{1} - \mathbb{K}_m$ describes the dual process $\tilde{\mathbf{x}}(\tau)$.

Theorem 2.1. *The determinantal point process $\tilde{\eta}(\tau, x)$ on $\{\tau_1, \ldots, \tau_p\} \times \mathbb{R}$, $\tau_i \in (-t, t)$, defined by the two groups of non-intersecting walkers, starting and ending $2m+1$ apart, at times $-t$ and t respectively, has gap probabilities on any compact set $E \subset \{\tau_1, \ldots, \tau_p\} \times \mathbb{R}$ given by*

$$\mathbb{P}(\tilde{\eta}(\mathbb{1}_E) = 0) = \det(\mathbb{1} - \widetilde{\mathbb{K}}_m^{\mathrm{ext}})_{L^2(E)}, \tag{2.4}$$

where the kernel $\widetilde{\mathbb{K}}_m^{\mathrm{ext}}$ is given by

$$\begin{aligned}
&\frac{e^{2t_2}}{e^{2t_1}} \widetilde{\mathbb{K}}_m^{\mathrm{ext}}(t_1, x_1; t_2, x_2) = -\mathbb{1}_{[t_2<t_1]} I_{|x_1-x_2|}(2(t_2 - t_1)) \\
&- \frac{V_m}{(2\pi \mathrm{i})^2} \oint_{\Gamma_0} dz \oint_{\Gamma_{0,z}} dw \frac{e^{t(z-z^{-1})}}{e^{t(w-w^{-1})}} \frac{e^{-t_1(z+z^{-1})}}{e^{-t_2(w+w^{-1})}} \frac{w^{x_2-m-1}}{z^{x_1-m}} \frac{H_{2m+1}(w) H_{2m+1}(z^{-1})}{z - w} \\
&- \frac{V_m}{(2\pi \mathrm{i})^2} \oint_{\Gamma_0} dw \oint_{\Gamma_{0,w}} dz \frac{e^{t(w-w^{-1})}}{e^{t(z-z^{-1})}} \frac{e^{-t_1(z+z^{-1})}}{e^{-t_2(w+w^{-1})}} \frac{w^{x_2+m}}{z^{x_1+m+1}} \frac{H_{2m+1}(z) H_{2m+1}(w^{-1})}{w - z} \\
&- \mathbb{1}_{[x_1 \neq x_2]} \frac{V_m}{2\pi \mathrm{i}} \oint_{\Gamma_0} dz \frac{e^{(t_2-t_1)(z+z^{-1})}}{z^{x_1-x_2+1}} H_{2m+1}(z^{-1}) H_{2m+1}(z),
\end{aligned} \tag{2.5}$$

with $V_m := 1/(H_{2m+1}(0) H_{2m+2}(0))$. The function H_n is itself the Fredholm determinant on $\ell^2(\{n, n+1, \ldots\})$

$$H_n(z^{-1}) := \det(\mathbb{1} - K(z^{-1}))_{\ell^2(\{n,n+1,\ldots\})} \tag{2.6}$$

of the kernel

$$K(z^{-1})_{k,\ell} := \frac{(-1)^{k+\ell}}{(2\pi \mathrm{i})^2} \oint_{\Gamma_0} du \oint_{\Gamma_{0,u}} dv \frac{u^\ell}{v^{k+1}} \frac{1}{v - u} \frac{u - z}{v - z} \frac{e^{2t(u-u^{-1})}}{e^{2t(v-v^{-1})}}, \tag{2.7}$$

where Γ_0 is any anticlockwise simple loop enclosing 0 and similarly $\Gamma_{0,u}$ encircles the poles at 0 and u (but not z)[1]*.*

The extended kernel, governing the process $\tilde{\eta}(\tau, x)$, is given in terms of the the kernel $\widetilde{\mathbb{K}}_m(x_1, x_2) = \widetilde{\mathbb{K}}_m^{\mathrm{ext}}(0, x_1; 0, x_2)$, governing the distribution $\tilde{\eta}(0, x)$, by

$$\widetilde{\mathbb{K}}_m^{\mathrm{ext}}(t_1, x_1; t_2, x_2) = -\mathbb{1}_{[t_2<t_1]} \left(e^{(t_2-t_1)\mathcal{H}}\right)(x_1, x_2) + \left(e^{-t_1 \mathcal{H}} \widetilde{\mathbb{K}}_m e^{t_2 \mathcal{H}}\right)(x_1, x_2), \tag{2.8}$$

where $\mathcal{H}$ is the discrete Laplacian

$$(\mathcal{H}f)(x) = f(x+1) + f(x-1) - 2f(x). \tag{2.9}$$

[1]For any set of points S, the notation $\oint_{\Gamma_S} dz f(z)$ means that the integration path goes anticlockwise around the points in S but does not include any other poles of f.

Remark that the transition probability of (2.1), defined for $t \geq 0$, can be written as $p_t(x, y) = e^{t\mathcal{H}}\mathbb{1}(x, y) =: e^{t\mathcal{H}}(x, y)$. Here, $\mathbb{1}$ denotes the identity operator on $\mathbb{Z}$, i.e., $\mathbb{1}(x, y) = 1$ if $x = y$ and $\mathbb{1}(x, y) = 0$ if $x \neq y$.

The formula for the kernel $\widetilde{\mathbb{K}}_m(t_1, x_1; t_2, x_2)$ (that is at $t_i = 0$) in Theorem 2.1 will be established in Section 3, whereas the one for $\widetilde{\mathbb{K}}_m^{\mathrm{ext}}$ will be shown in Section 5. In Sections 4 and 5, it will be shown that both kernels $\widetilde{\mathbb{K}}_m(x, y)$ and $\widetilde{\mathbb{K}}_m^{\mathrm{ext}}(t_1, x_2; t_2, x_2)$ have a representation, whose constituents can be expressed in terms of Bessel functions; see the expression (4.14) and the time-dependent kernel (5.26), derived from (4.14), via the recipe (2.8). Also, note that the kernel $K(z^{-1})$ is a rank-one perturbation of the kernel $K(0)$, whose Fredholm determinant

$$H_n(0) = \det(\mathbb{1} - K(0))_{\ell^2(\{n,n+1,\dots\})} \tag{2.10}$$

is the distribution of the longest increasing subsequence of a random permutation in the Poissonized version, or, equivalently, it is the height function in the polynuclear growth (PNG) model [10,41]. In the scaling limit, considered in Section 6, $H_n(0)$ will converge to the Tracy-Widom distribution F_2.

To study the limiting behavior, when $m, t \to \infty$, consider first the system of non-intersecting random walks starting at time $-t$ and ending at positions $\{\dots, -m-2, -m-1\}$ at time t. This is, up to a shift by $m + 1$, the multi-layer PNG model studied by Prähofer and Spohn in [41]. Their work shows that the top random walk at time $\tau = 0$ has fluctuations around $x = -m+2t$ of order $t^{1/3}$. By symmetry, if one considers only the non-intersecting random walks starting and ending at position $\{m+1, m+2, \dots\}$, the bottom random walk at time $\tau = 0$ fluctuates around $x = m - 2t$ also in the spatial scale $t^{1/3}$.

The top and bottom random walks interact if the proportion of deleted configurations, due to interaction, is non-zero. This happens when $m = 2t$ to leading order in t. The first scaling where interaction is relevant is given by $m = 2t+\sigma t^{1/3}$. The parameter σ modulates the strength of interaction of the two sets of non-intersecting random walks. In the extreme cases $\sigma \to \infty$, we clearly (by a simple probabilistic argument) go back to the situation of two independents PNG models, thus the top of the lower walks and the bottom of the upper walks are governed by the Airy$_2$ process [41]. On the other hand, when $\sigma \to -\infty$, one expects to see a point process governed by the sine kernel or the Pearcey process. Moreover, locally the paths will looks like random walks, so the exponents in the scaling for time and space are in a ratio 2:1. Thus, we set the scaling[2]

$$m = 2t + \sigma t^{1/3}, \quad x_i = \xi_i t^{1/3}, \quad t_i = s_i t^{2/3}, \quad i = 1, 2. \tag{2.11}$$

[2]We do not write explicitly the integer parts, since in the $t \to \infty$ it is irrelevant.

Also note that for each time $-t < \tau < t$, the density of particles has its support on two semi-infinite intervals, whose boundary, as a function of τ, describes two curves, which at $\tau = 0$ form a *tacnode*. The purpose of Theorem 2.2 is to describe the fluctuations of the random walks in the $t \to \infty$ limit in the neighborhood of $(x, \tau) = (0, 0)$, but in the new space-time scale, given by (2.11).

In order to state the second main result, define the standard Airy kernel,

$$K_{\mathrm{Ai}}(\xi_1, \xi_2) := \int_0^\infty d\lambda \, \mathrm{Ai}(\xi_1 + \lambda)\mathrm{Ai}(\xi_2 + \lambda). \tag{2.12}$$

and the function $\mathcal{Q}(\kappa)$, already appearing in [48],

$$\mathcal{Q}(\kappa) := [(\mathbb{1} - \chi_{\tilde{\sigma}} K_{\mathrm{Ai}} \chi_{\tilde{\sigma}})^{-1} \chi_{\tilde{\sigma}} \mathrm{Ai}](\kappa), \quad \text{with } \tilde{\sigma} := 2^{2/3}\sigma, \tag{2.13}$$

and where $\chi_a(x) = \mathbb{1}_{[x>a]}$. We further set

$$\mathrm{Ai}^{(s)}(\xi) := e^{\xi s + \frac{2}{3}s^3} \mathrm{Ai}(\xi + s^2), \tag{2.14}$$

which equals to the standard Airy function $\mathrm{Ai}(\xi)$, when $s = 0$, and define the functions

$$\begin{aligned} \mathcal{A}(s, \xi) &:= \mathrm{Ai}^{(s)}(\sigma - \xi) + \int_{\tilde{\sigma}}^\infty d\kappa \int_0^\infty d\alpha \, \mathcal{Q}(\kappa)\mathrm{Ai}(\kappa + \alpha)\mathrm{Ai}^{(s)}(2^{1/3}\alpha + \sigma - \xi), \\ \mathcal{B}(s, \xi) &:= \int_{\tilde{\sigma}}^\infty d\kappa \, \mathcal{Q}(\kappa)\mathrm{Ai}^{(s)}(2^{1/3}\kappa - \sigma + \xi), \end{aligned} \tag{2.15}$$

and

$$\begin{aligned} \mathcal{C}(s, \xi) &:= 2^{-1/3} \int_{\tilde{\sigma}}^\infty d\kappa \, \mathcal{Q}(\kappa) \Big[\mathrm{Ai}^{(2^{-2/3}s)}(\kappa + 2^{-1/3}\xi) \\ &+ \int_{\tilde{\sigma}}^\infty d\lambda \, \mathcal{Q}(\lambda) \int_0^\infty d\alpha \, \mathrm{Ai}(\alpha + \lambda)\mathrm{Ai}^{(2^{-2/3}s)}(\alpha + \kappa + 2^{-1/3}\xi) \Big] + (\xi \leftrightarrow -\xi), \end{aligned} \tag{2.16}$$

where with $(\xi \leftrightarrow -\xi)$ we mean the same expression with ξ replaced by $-\xi$. Finally, we define two Laplace transforms $\hat{\mathcal{P}}(u)$ and $\hat{\mathcal{Q}}(u)$:

$$\begin{aligned} \hat{\mathcal{Q}}(u) &:= \int_{\tilde{\sigma}}^\infty d\kappa \, \mathcal{Q}(\kappa) e^{\kappa u 2^{1/3}}, \\ \hat{\mathcal{P}}(u) &:= -\int_0^\infty d\kappa \, e^{-\kappa u 2^{1/3}} \int_{\tilde{\sigma}}^\infty d\mu \, \mathcal{Q}(\mu)\mathrm{Ai}(\mu + \kappa). \end{aligned} \tag{2.17}$$

Theorem 2.2. *Near the tacnode appears a new determinantal process on $\{s_1, \dots, s_p\} \times \mathbb{R}$, the tacnode process $\mathcal{T}$, whose gap probabilities on any compact set $E \subset \{s_1, \dots, s_p\} \times \mathbb{R}$ are given by*

$$\mathbb{P}(\mathcal{T}(\mathbb{1}_E) = 0) = \det(\mathbb{1} - \mathcal{K}^{\mathrm{ext}})_{L^2(E)}. \tag{2.18}$$

The kernel $\mathcal{K}^{\mathrm{ext}}$ is the limit of $\widetilde{\mathbb{K}}_m^{\mathrm{ext}}$ under the scaling (2.11),

$$\mathcal{K}^{\mathrm{ext}}(s_1, \xi_1; s_2, \xi_2) := \lim_{t\to\infty} \frac{(-1)^{x_2} e^{4t_2}}{(-1)^{x_1} e^{4t_1}}\, t^{1/3} \widetilde{\mathbb{K}}_m^{\mathrm{ext}}(t_1, x_1; t_2, x_2), \tag{2.19}$$

where the convergence is uniform for ξ_1, ξ_2 and s_1, s_2 in bounded sets. The kernel $\mathcal{K}^{\mathrm{ext}}$ has the following representations:

$$\begin{aligned}\mathcal{K}^{\mathrm{ext}}(s_1, \xi_1; s_2, \xi_2) = &-\frac{\mathbb{1}_{[s_2<s_1]}}{\sqrt{4\pi(s_1 - s_2)}} \exp\left(-\frac{(\xi_1 - \xi_2)^2}{4(s_1 - s_2)}\right) + \mathcal{C}(s_1 - s_2, \xi_1 - \xi_2)\\
&+ \int_0^\infty d\gamma \Big(\mathcal{A}(s_1, \xi_1 - \gamma)\mathcal{A}(-s_2, \xi_2 - \gamma) + \mathcal{A}(s_1, -\xi_1 - \gamma)\mathcal{A}(-s_2, -\xi_2 - \gamma)\\
&\quad -\mathcal{A}(s_1, \xi_1 - \gamma)\mathcal{B}(-s_2, \xi_2 - \gamma) - \mathcal{A}(s_1, -\xi_1 - \gamma)\mathcal{B}(-s_2, -\xi_2 - \gamma)\\
&\quad -\mathcal{B}(s_1, \xi_1 - \gamma)\mathcal{A}(-s_2, \xi_2 - \gamma) - \mathcal{B}(s_1, -\xi_1 - \gamma)\mathcal{A}(-s_2, -\xi_2 - \gamma)\Big)\\
&- \int_{-\infty}^0 d\gamma \Big(\mathcal{B}(s_1, \xi_1 - \gamma)\mathcal{B}(-s_2, \xi_2 - \gamma) + \mathcal{B}(s_1, -\xi_1 - \gamma)\mathcal{B}(-s_2, -\xi_2 - \gamma)\Big),\end{aligned} \tag{2.20}$$

as well as

$$\begin{aligned}\mathcal{K}^{\mathrm{ext}}(s_1, \xi_1; s_2, \xi_2) = &-\frac{\mathbb{1}_{[s_2<s_1]}}{\sqrt{4\pi(s_1 - s_2)}} \exp\left(-\frac{(\xi_1 - \xi_2)^2}{4(s_1 - s_2)}\right) + \mathcal{C}(s_1 - s_2, \xi_1 - \xi_2)\\
&+ \frac{1}{(2\pi \mathrm{i})^2} \int_{\delta+\mathrm{i}\mathbb{R}} du \int_{-\delta+\mathrm{i}\mathbb{R}} dv\, \frac{e^{\frac{u^3}{3} - \sigma u}}{e^{\frac{v^3}{3} - \sigma v}} \frac{e^{s_1 u^2}}{e^{s_2 v^2}} \left(\frac{e^{\xi_1 u}}{e^{\xi_2 v}} + \frac{e^{-\xi_1 u}}{e^{-\xi_2 v}}\right) \frac{(1 - \hat{\mathcal{P}}(u))(1 - \hat{\mathcal{P}}(-v))}{u - v}\\
&- \frac{1}{(2\pi \mathrm{i})^2} \int_{2\delta+\mathrm{i}\mathbb{R}} du \int_{\delta+\mathrm{i}\mathbb{R}} dv\, \frac{e^{\frac{u^3}{3} - \sigma u}}{e^{-\frac{v^3}{3} - \sigma v}} \frac{e^{s_1 u^2}}{e^{s_2 v^2}} \left(\frac{e^{\xi_1 u}}{e^{\xi_2 v}} + \frac{e^{-\xi_1 u}}{e^{-\xi_2 v}}\right) \frac{(1 - \hat{\mathcal{P}}(u))\hat{\mathcal{Q}}(-v)}{u - v}\\
&- \frac{1}{(2\pi \mathrm{i})^2} \int_{-\delta+\mathrm{i}\mathbb{R}} du \int_{-2\delta+\mathrm{i}\mathbb{R}} dv\, \frac{e^{-\frac{u^3}{3} - \sigma u}}{e^{\frac{v^3}{3} - \sigma v}} \frac{e^{s_1 u^2}}{e^{s_2 v^2}} \left(\frac{e^{\xi_1 u}}{e^{\xi_2 v}} + \frac{e^{-\xi_1 u}}{e^{-\xi_2 v}}\right) \frac{(1 - \hat{\mathcal{P}}(-v))\hat{\mathcal{Q}}(u)}{u - v}\\
&+ \frac{1}{(2\pi \mathrm{i})^2} \int_{-\delta+\mathrm{i}\mathbb{R}} du \int_{\delta+\mathrm{i}\mathbb{R}} dv\, \frac{e^{-\frac{u^3}{3} - \sigma u}}{e^{-\frac{v^3}{3} - \sigma v}} \frac{e^{s_1 u^2}}{e^{s_2 v^2}} \left(\frac{e^{\xi_1 u}}{e^{\xi_2 v}} + \frac{e^{-\xi_1 u}}{e^{-\xi_2 v}}\right) \frac{\hat{\mathcal{Q}}(u)\hat{\mathcal{Q}}(-v)}{u - v}.\end{aligned} \tag{2.21}$$

The form (2.20) of the limiting extended kernel in Theorem 2.2 will be shown in Section 6, whereas a sketch of the proof of its double integral representation (2.21) will be given in Section 7.

In the preprint [21] the analogue problem for Brownian Motion will be analyzed with the Riemann-Hilbert approach to multiple orthogonal polynomials. It would be interesting to see how to relate the two formulas (which we expect to be the equivalent).

3 Finite system at $\tau = 0$

In this section we will prove Theorem 2.1 for $t_1 = t_2 = 0$. Consider a continuous time random walk in $\mathbb{Z}$ with jumps ± 1, which occur independently with rate 1, i.e., the waiting times of the up- and down-jumps are independent and exponentially distributed with mean 1. Thus, the number of up-jumps (and similarly down-jumps) during the time interval $[0,t]$ is Poisson distributed,

$$\mathbb{P}(k \text{ up-jumps during } [0,t]) = e^{-t}\frac{t^k}{k!}. \tag{3.1}$$

As will be shown, the transition probability $p_t(x,y)$ of going from x to y during a time interval of length t is given by

$$p_t(x,y) = e^{-2t} I_{|x-y|}(2t), \tag{3.2}$$

where I_n is the modified Bessel function of degree n (see [1]). To prove (3.2), first notice that by symmetry, it is enough to consider $y - x \geq 0$. To go from x to y, the process must perform k steps to the left and $k+y-x$ steps to the right. Since the moment, at which the left or right steps occur, is independent of whether it is a left or a right step, on may assume the process doing first k steps to the left and then $k+y-x$ steps to the right. By the strong Markov property of the random walk and the independence of the jumps,

$$\begin{aligned} p_t(x,y) &= \sum_{k\geq 0} \mathbb{P}\left(\left\{ \begin{array}{l} k+y-x \text{ up-steps and} \\ k \text{ down-steps} \end{array} \right\} \text{during time } t \right) \\ &= e^{-2t} \sum_{k\geq 0}^{\infty} \frac{t^k}{k!} \frac{t^{y-x+k}}{(y-x+k)!} = e^{-2t} I_{|x-y|}(2t). \end{aligned} \tag{3.3}$$

The modified Bessel function has the following expressions (for $n \in \mathbb{Z}$)

$$I_n(2t) = \frac{1}{2\pi \mathrm{i}} \oint_{S^1} \frac{dz}{z} e^{t(z+z^{-1})} z^{\pm n} = \sum_{k=0}^{\infty} \frac{t^k}{k!} \frac{t^{k+|n|}}{(k+|n|)!}, \tag{3.4}$$

with $S^1 = \{z \in \mathbb{C} | |z| = 1\}$.

Consider now $n = 2m+1$ ($m \in \mathbb{N}$) continuous time random walks starting from $-m, -m+1, \ldots, m$ at time $\tau = -t$, returning at the starting positions at time $\tau = t$, and *conditioned not to intersect*. Denote by $x_k(\tau)$ the position at time τ of the random walk which started from $m+1-k$ (i.e., the kth highest one), see Figure 2 for an illustration with $m = 2$.

The probability at time $\tau = 0$ is easily obtained by the Karlin-McGregor formula [31], namely

$$\begin{aligned}
&\mathbb{P}\left(\bigcap_{k=1}^{2m+1}\{x_k(0) = y_k\}\Bigg|\bigcap_{k=1}^{2m+1}\{x_k(t) = x_k(-t) = m+1-k\}\right) \\
&= \text{const} \times \det\left[p_t(m+1-i, y_j)\right]_{1\le i,j\le 2m+1} \det\left[p_t(y_i, m+1-j)\right]_{1\le i,j\le 2m+1} \\
&= \text{const} \times \left(\det\left[I_{y_i+j-1-m}(2t)\right]_{1\le i,j\le 2m+1}\right)^2.
\end{aligned} \tag{3.5}$$

It is well known by [13] that the process above

$$\mathbf{x}(\tau) := \{x_k(\tau), 1 \le k \le 2m+1\}, \quad \tau \in [-t, t], \tag{3.6}$$

with a measure of this form gives rise to a determinantal point process (random point measure)

$$\eta = \sum_{k=1}^{2m+1} \delta_{x_k(0)} \tag{3.7}$$

with a certain kernel $\mathbb{K}_m(x, y)$, to be computed in Theorem 3.1.

Instead of the process $\mathbf{x}(\tau)$, we shall analyze its complementary (dual) process, which we denote by

$$\tilde{\mathbf{x}}(\tau) = \{\tilde{x}_k(\tau), k \in \mathbb{Z} \setminus \{1 \le k \le 2m+1\}\}, \quad t \in [-t, t]. \tag{3.8}$$

If $\mathbf{x}$ denotes the trajectories of the $2m+1$ particles, then let $\tilde{\mathbf{x}}$ denote the trajectories of the holes, obtained by the particle-hole transformation, see Figures 1 and 3.

The reason for starting with the process $\mathbf{x}$ is that the Karlin-McGregor formula applies to a finite number of paths, while $\tilde{\mathbf{x}}$ has an infinite number of paths. By the complementation principle in the Appendix of [17], the dual point process at $\tau = 0$,

$$\tilde{\eta} = \sum_k \delta_{\tilde{x}_k(0)}, \tag{3.9}$$

is also determinantal with correlation kernel

$$\widetilde{\mathbb{K}}_m(x, y) = \delta_{x,y} - \mathbb{K}_m(x, y). \tag{3.10}$$

First of all, we compute the kernel $\mathbb{K}_m(x, y)$ in a form which will be suitable for asymptotic analysis.

Theorem 3.1. *The point processes η and $\tilde{\eta}$, defined in (3.7) and (3.9), are determinantal with correlation kernel $\mathbb{K}_m$ and $\widetilde{\mathbb{K}}_m$ given below. Thus, for any finite subset $E \subset \mathbb{Z}$, the gap probability of E is given by*

$$\mathbb{P}\left(\eta(\mathbb{1}_E) = 0\right) = \det(\mathbb{1} - \mathbb{K}_m)_{\ell^2(E)}, \quad \mathbb{P}\left(\tilde{\eta}(\mathbb{1}_E) = 0\right) = \det(\mathbb{1} - \widetilde{\mathbb{K}}_m)_{\ell^2(E)}, \tag{3.11}$$

with kernels $\mathbb{K}_m(x,y)$ and $\widetilde{\mathbb{K}}_m(x,y)$, invariant[3] *under the involution $(x,y) \leftrightarrow (-y,-x)$, namely*

$$\begin{aligned}
\mathbb{K}_m(x,y) = {} & \frac{V_m}{(2\pi \mathrm{i})^2} \oint_{\Gamma_0} dz \oint_{\Gamma_{0,z}} dw \frac{e^{t(z-z^{-1})}}{e^{t(w-w^{-1})}} \frac{w^{y-m-1}}{z^{x-m}} \frac{H_{2m+1}(w) H_{2m+1}(z^{-1})}{z-w} \\
& + \frac{V_m}{(2\pi \mathrm{i})^2} \oint_{\Gamma_0} dw \oint_{\Gamma_{0,w}} dz \frac{e^{t(w-w^{-1})}}{e^{t(z-z^{-1})}} \frac{w^{y+m}}{z^{x+m+1}} \frac{H_{2m+1}(z) H_{2m+1}(w^{-1})}{w-z} \\
& + \frac{V_m}{2\pi \mathrm{i}} \oint_{\Gamma_0} dz \frac{1}{z^{x-y+1}} H_{2m+1}(z^{-1}) H_{2m+1}(z),
\end{aligned} \tag{3.12}$$

and

$$\begin{aligned}
\widetilde{\mathbb{K}}_m(x,y) = {} & - \frac{V_m}{(2\pi \mathrm{i})^2} \oint_{\Gamma_0} dz \oint_{\Gamma_{0,z}} dw \frac{e^{t(z-z^{-1})}}{e^{t(w-w^{-1})}} \frac{w^{y-m-1}}{z^{x-m}} \frac{H_{2m+1}(w) H_{2m+1}(z^{-1})}{z-w} \\
& - \frac{V_m}{(2\pi \mathrm{i})^2} \oint_{\Gamma_0} dw \oint_{\Gamma_{0,w}} dz \frac{e^{t(w-w^{-1})}}{e^{t(z-z^{-1})}} \frac{w^{y+m}}{z^{x+m+1}} \frac{H_{2m+1}(z) H_{2m+1}(w^{-1})}{w-z} \\
& - \mathbb{1}_{[x \neq y]} \frac{V_m}{2\pi \mathrm{i}} \oint_{\Gamma_0} dz \frac{1}{z^{x-y+1}} H_{2m+1}(z^{-1}) H_{2m+1}(z),
\end{aligned} \tag{3.13}$$

where $V_m = 1/(H_{2m+1}(0) H_{2m+2}(0))$. The function H_n itself is a Fredholm determinant on $\ell^2(\{n, n+1, \ldots\})$

$$H_n(z^{-1}) := \det(\mathbb{1} - K(z^{-1}))_{\ell^2(\{n,n+1,\ldots\})} \tag{3.14}$$

of the kernel

$$K(z^{-1})_{k,\ell} := \frac{(-1)^{k+\ell}}{(2\pi \mathrm{i})^2} \oint_{\Gamma_0} du \oint_{\Gamma_{0,u}} dv \frac{u^\ell}{v^{k+1}} \frac{1}{v-u} \frac{u-z}{v-z} \frac{e^{2t(u-u^{-1})}}{e^{2t(v-v^{-1})}}, \tag{3.15}$$

where Γ_0 is any anticlockwise simple loop enclosing 0 and similarly $\Gamma_{0,u}$ encircles 0 and u only (hence not z).

[3] As it should from the geometry of the problem! The involution interchanges the two double integrals in (3.12), as is seen from renaming $w \leftrightarrow z$ in the second double integral; also the third term, the single integral, only depends on $|x-y|$, as is seen from $z \to z^{-1}$.

Proof of Theorem 3.1.

Step 1: Computing the kernel $\mathbb{K}_m(x,y)$, from the Karlin-McGregor formula (3.5): It is well known by [13] that a measure of the form (3.5) implies that the point process (random point measure) η, as in (3.7), is determinantal with correlation kernel

$$\mathbb{K}_m(x,y) = \sum_{k,\ell=1}^{2m+1} \varphi_k(y)[A^{-1}]_{k,\ell}\, \varphi_\ell(x), \quad x,y \in \mathbb{Z}, \tag{3.16}$$

where

$$\varphi_k(x) = I_{x+k-1-m}(2t) \tag{3.17}$$

and A is the $(2m+1)\times(2m+1)$ matrix with entries

$$[A]_{k,\ell} \equiv \langle \varphi_k, \varphi_\ell \rangle = \sum_{x\in\mathbb{Z}} \varphi_k(x)\varphi_\ell(x). \tag{3.18}$$

Using (3.17), the entries of the $(2m+1)\times(2m+1)$ matrix A, as in (3.18), are given by

$$\begin{aligned} A_{k,\ell} &= \sum_{x\in\mathbb{Z}} \varphi_k(x)\varphi_\ell(x) = \sum_{x\geq 0} \varphi_k(x)\varphi_\ell(x) + \sum_{x<0} \varphi_k(x)\varphi_\ell(x) \\ &= \sum_{x\geq 0} \frac{1}{(2\pi \mathrm{i})^2} \oint_{\Gamma_0} dz \oint_{\Gamma_0} dw \frac{e^{t(z+z^{-1})}e^{t(w+w^{-1})}}{z^k w^\ell} \frac{1}{(zw)^{x-m}} \\ &+ \sum_{x<0} \frac{1}{(2\pi \mathrm{i})^2} \oint_{\Gamma_0} dz \oint_{\Gamma_0} dw \frac{e^{t(z+z^{-1})}e^{t(w+w^{-1})}}{z^k w^\ell} \frac{1}{(zw)^{x-m}}. \end{aligned} \tag{3.19}$$

In the first integrals, we deform the paths to $|z|=1$ and $|w|=R>1$. Then we take the sum inside the integrals and use $\sum_{x\geq 0}(zw)^{-x} = wz/(wz-1)$. Similarly, in the second integrals, we deform the paths as $|z|=1$ and $|w|=1/R<1$ and use $\sum_{x<0}(zw)^{-x} = -wz/(wz-1)$. This leads to

$$\begin{aligned} A_{k,\ell} &= \frac{1}{(2\pi \mathrm{i})^2} \oint_{|z|=1} dz \oint_{|w|=R} dw \frac{e^{t(z+z^{-1})}e^{t(w+w^{-1})}}{z^{k-m} w^{\ell-m}} \frac{wz}{wz-1} \\ &- \frac{1}{(2\pi \mathrm{i})^2} \oint_{|z|=1} dz \oint_{|w|=1/R} dw \frac{e^{t(z+z^{-1})}e^{t(w+w^{-1})}}{z^{k-m} w^{\ell-m}} \frac{wz}{wz-1} \\ &= \frac{1}{2\pi \mathrm{i}} \oint_{|z|=1} dz \frac{e^{2t(z+z^{-1})}}{z^{k-\ell+1}} = I_{k-\ell}(4t), \end{aligned} \tag{3.20}$$

since for any value of z, the two integrals differ only by the residue[4] at $w=1/z$. However, doing the asymptotics of the kernel $\mathbb{K}_m(x,y)$ with this choice of basis and thus with this A^{-1} seems to be hopeless.

[4]This residue argument will reappear later in (3.40).

Step 2: Changing the basis $\varphi_k \mapsto \psi_k$, such that $A \mapsto \mathbb{1}$ in the kernel $\mathbb{K}_m(x,y)$, i.e., so that $\mathbb{K}_m(x,y) = \sum_{k=1}^{2m+1} \psi_k(x)\psi_k(y)$. Replace the basis $(\varphi_k(x))_{k=1,\dots,2m+1}$ with an orthonormal basis $(\psi_k(x))_{k=1,\dots,2m+1}$ with respect to the measure $d\rho_t(z) := \frac{dz}{2\pi i z} e^{t(z+z^{-1})}$ (generating the same vector space, i.e., $\det(\varphi_k(x_j))_{1\le k,j\le n} = \text{const} \times \det(\psi_k(x_j))_{1\le k,j\le n}$ so that the measure (3.5) has the same form but with $A = \mathbb{1}$). More precisely, we shall search for polynomials P_k of degree k such that

$$\psi_k(x) = \oint_{S^1} \frac{d\rho_t(z)}{z^{x-m}} P_{k-1}(z^{-1}) = \oint_{S^1} d\rho_t(w) w^{x-m} P_{k-1}(w), \quad 1 \le k \le 2m+1, \tag{3.21}$$

satisfies, using the same argument as in (3.20),

$$\begin{aligned} \delta_{k,l} = \langle \psi_k, \psi_\ell \rangle &= \sum_{x\in\mathbb{Z}} \oint_{\Gamma_0} d\rho_t(z) \oint_{\Gamma_0} d\rho_t(w) (zw)^{x-m} P_{k-1}(z) P_{\ell-1}(w) \\ &= \oint_{S^1} d\rho_{2t}(z) P_{k-1}(z) P_{\ell-1}(z^{-1}) =: \langle\!\langle P_{k-1}, P_{\ell-1} \rangle\!\rangle, \end{aligned} \tag{3.22}$$

thus defining a new inner-product $\langle\!\langle \;,\; \rangle\!\rangle$ on the circle $S^1 = \{z \in \mathbb{C} \mid |z| = 1\}$. So it suffices to find an orthonormal basis of polynomials on the circle for the weight $d\rho_{2t}(z)$. A classical expression for the polynomial $P_k(z)$ is (see e.g. [46]):

$$P_k(z) = \frac{1}{\sqrt{\det m_k \cdot \det m_{k+1}}} \det \begin{pmatrix} & 1 \\ [\mu_{i,j}]_{\substack{0\le i\le k \\ 0\le j\le k-1}} & z \\ & \vdots \\ & z^k \end{pmatrix}, \tag{3.23}$$

where $m_k = [\mu_{i,j}]_{0\le i,j\le k-1}$ and

$$\mu_{i,j} := \langle\!\langle z^i, z^j \rangle\!\rangle = \oint_{S^1} d\rho_{2t}(z) z^{i-j} = I_{i-j}(4t). \tag{3.24}$$

Hence the $P_k(z)$ are polynomials of z with real coefficients. Orthonormal polynomials on the circle satisfy a Christoffel-Darboux-type formula, due to Szegő; see [43]. Namely, with the notation $P_n^*(z) = z^n \overline{P(\bar{z}^{-1})}$ and further using the reality of the coefficients, one obtains for $z, w \in S^1$,

$$\begin{aligned} \sum_{\ell=0}^{n-1} P_\ell(z^{-1}) P_\ell(w) = \sum_{\ell=0}^{n-1} \overline{P_\ell(z)} P_\ell(w) &= \frac{\overline{P_n^*(z)} P_n^*(w) - \overline{P_n(z)} P_n(w)}{1 - \bar{z} w}, \\ &= \frac{\overline{z^n \overline{P_n(\bar{z}^{-1})}} w^n \overline{P_n(\bar{w}^{-1})} - \overline{P_n(z)} P_n(w)}{1 - w/z} \\ &= \frac{z^{-n} P_n(z) w^n P_n(w^{-1}) - P_n(z^{-1}) P_n(w)}{1 - w/z}. \end{aligned} \tag{3.25}$$

Step 3: Expressing the polynomials $P_n(z)$ in terms of the Fredholm determinant $H_n(z^{-1})$, as in (3.14). In order to do this, one first introduces the inner-product

$$\langle f, g\rangle_{\mathbf{t},\mathbf{s}} := \frac{1}{2\pi\mathrm{i}}\oint_{S^1}\frac{du}{u}f(u)g(u^{-1})e^{\sum_{j=1}^{\infty}(t_ju^j-s_ju^{-j})}, \tag{3.26}$$

upon setting $\mathbf{t} := (t_1, t_2, \ldots) \in \mathbb{C}^\infty$ and $\mathbf{s} := (s_1, s_2, \ldots) \in \mathbb{C}^\infty$. It was shown in [7, 8] (see also the lecture notes [51]) that the functions[5]

$$\begin{aligned} p_n^{(1)}(\mathbf{t},\mathbf{s};z) &:= z^n\frac{\tau_n(\mathbf{t}-[z^{-1}],\mathbf{s})}{\sqrt{\tau_n(\mathbf{t},\mathbf{s})\tau_{n+1}(\mathbf{t},\mathbf{s})}} \\ p_n^{(2)}(\mathbf{t},\mathbf{s};z) &:= z^n\frac{\tau_n(\mathbf{t},\mathbf{s}+[z^{-1}])}{\sqrt{\tau_n(\mathbf{t},\mathbf{s})\tau_{n+1}(\mathbf{t},\mathbf{s})}} \end{aligned} \tag{3.27}$$

are bi-orthonormal polynomials with regard to the inner-product (3.26). In the formulae above, the $\tau_n(\mathbf{t},\mathbf{s})$ are 2-Toda τ-functions and are defined as a Toeplitz determinant, which is also expressible as a Fredholm determinant of the kernel (3.29) below, using the Borodin-Okounkov identity [16]. We obtain

$$\begin{aligned} \tau_n(\mathbf{t},\mathbf{s}) &:= \det\left[\frac{1}{2\pi\mathrm{i}}\oint_{S^1}\frac{du}{u}u^{k-\ell}e^{\sum_{j=1}^{\infty}(t_ju^j-s_ju^{-j})}\right]_{1\le k,\ell\le n} \\ &= Z(\mathbf{t},\mathbf{s})\det\left(\mathbb{1}-\mathbf{K}(\mathbf{t},\mathbf{s})\right)_{\ell^2(\{n,n+1,\ldots\})}, \quad Z(\mathbf{t},\mathbf{s}) := e^{-\sum_{j=1}^{\infty}j\,t_js_j}, \end{aligned} \tag{3.28}$$

where the kernel $\mathbf{K}(\mathbf{t},\mathbf{s})$ is given by

$$\mathbf{K}(\mathbf{t},\mathbf{s})_{k,\ell} := \frac{1}{(2\pi\mathrm{i})^2}\oint_{\Gamma_0}du\oint_{\Gamma_{0,u}}dv\frac{u^\ell}{v^{k+1}}\frac{1}{v-u}\frac{e^{\sum_{j=1}^{\infty}(t_jv^{-j}+s_jv^j)}}{e^{\sum_{j=1}^{\infty}(t_ju^{-j}+s_ju^j)}}. \tag{3.29}$$

The coefficients t_j, s_j have to be such that the expression $\sum_{j=1}^{\infty}(t_ju^j - s_ju^{-j})$ appearing in the exponent of (3.28) is analytic in the annulus $\rho < |z| < \rho^{-1}$ for $0 < \rho < 1$. Then, the Borodin-Okounkov identity (3.28) gives a kernel $\mathbf{K}(\mathbf{t},\mathbf{s})$, with contours given by $|u| = |v|^{-1} = \rho'$, with $0 < \rho < \rho' < 1$. Using Cauchy's Theorem the contours may be deformed to any circle of radius $0 < \rho < 1$. Then, using $\sum_{j=1}^{\infty}(v/z)^j/j = -\ln(1-v/z)$ (for $|v/z| < 1$) we obtain

$$\mathbf{K}(\mathbf{t},\mathbf{s}+[z^{-1}])_{k,\ell} = \frac{1}{(2\pi\mathrm{i})^2}\oint_{\Gamma_0}du\oint_{\Gamma_{0,u}}dv\frac{u^\ell}{v^{k+1}}\frac{1}{v-u}\frac{1-\frac{u}{z}}{1-\frac{v}{z}}\frac{e^{\sum_{j=1}^{\infty}(t_jv^{-j}+s_jv^j)}}{e^{\sum_{j=1}^{\infty}(t_ju^{-j}+s_ju^j)}} \tag{3.30}$$

[5]For $\alpha \in \mathbb{C}$, one defines $[\alpha] = \left(\alpha, \frac{\alpha^2}{2}, \frac{\alpha^3}{3}, \ldots\right) \in \mathbb{C}^\infty$.

and

$$Z(\mathbf{t},\mathbf{s}+[z^{-1}]) = e^{-\sum_{j=1}^{\infty} j\, t_j (s_j + z^{-j}/j)} = Z(\mathbf{t},\mathbf{s}) e^{-\sum_{j=1}^{\infty} t_j z^{-j}}. \tag{3.31}$$

We now specialize all this to the locus

$$\mathcal{L} = \left\{ \begin{array}{l} \mathbf{t} = (2t, 0, 0, ...) \\ \mathbf{s} = (-2t, 0, 0, ...) \end{array} \right\}. \tag{3.32}$$

On this locus, one checks that $Z(\mathbf{t},\mathbf{s})\big|_{\mathcal{L}} = e^{4t^2}$, that $\mathbf{K}(\mathbf{t},\mathbf{s})$ and its translate, restricted to the locus $\mathcal{L}$, are closely related to the kernel $K(z^{-1})$ defined in (3.15)[6]

$$\begin{aligned} \mathbf{K}(\mathbf{t},\mathbf{s})\big|_{\mathcal{L}} &\overset{\text{conj}}{=} K(0), \\ \mathbf{K}(\mathbf{t},\mathbf{s}+[z^{-1}])\big|_{\mathcal{L}} &\overset{\text{conj}}{=} K(z^{-1}), \end{aligned} \tag{3.33}$$

and that the restriction of $\tau_n(\mathbf{t},\mathbf{s})$ to $\mathcal{L}$ leads to the Fredholm determinant $H_n(z^{-1})$ as defined in (3.14):

$$\begin{aligned} \tau_n(\mathbf{t},\mathbf{s})\big|_{\mathcal{L}} &= H_n(0) Z(\mathbf{t},\mathbf{s})\big|_{\mathcal{L}} = e^{4t^2} H_n(0), \\ \tau_n(\mathbf{t},\mathbf{s}+[z^{-1}])\big|_{\mathcal{L}} &= H_n(z^{-1}) e^{-2t/z} Z(\mathbf{t},\mathbf{s})\big|_{\mathcal{L}} = H_n(z^{-1}) e^{4t^2 - 2t/z}. \end{aligned} \tag{3.34}$$

Moreover, the inner-product $\langle f, g\rangle_{\mathbf{t},\mathbf{s}}$ defined in (3.26) reduces to the inner-product $\langle\!\langle f, g\rangle\!\rangle$ defined in (3.22),

$$\langle f, g\rangle_{\mathbf{t},\mathbf{s}}\big|_{\mathcal{L}} = \frac{1}{2\pi \mathrm{i}} \oint_{S^1} \frac{du}{u} e^{2t(u+u^{-1})} f(u) g(u^{-1}) = \langle\!\langle f, g\rangle\!\rangle. \tag{3.35}$$

It follows that the bi-orthogonal functions for $\langle f(z), g(z)\rangle_{\mathbf{t},\mathbf{s}}$ restricted to the locus $\mathcal{L}$ coincide with the orthonormal polynomials defined by (3.22), which by (3.27), (3.34) and (3.33) yields:

$$P_n(z) = p_n^{(1)}(\mathbf{t},\mathbf{s};z)\big|_{\mathcal{L}} = p_n^{(2)}(\mathbf{t},\mathbf{s};z)\big|_{\mathcal{L}} = \frac{z^n e^{-2t/z} H_n(z^{-1})}{\sqrt{H_n(0) H_{n+1}(0)}}, \tag{3.36}$$

where

$$H_n(z^{-1}) = \det(\mathbb{1} - K(z^{-1}))_{\ell^2(\{n,n+1,...\})}, \tag{3.37}$$

with the kernel $K(z^{-1})$ as in (3.15); this follows from (3.33). The fact that the $p_n^{(1)}$ and $p_n^{(2)}$ are equal on the locus $\mathcal{L}$ is a consequence of the symmetry

[6]With $A \overset{\text{conj}}{=} B$ we mean that the two kernels A and B are conjugate kernels. In the present case, the conjugation factor is $(-1)^{k-\ell}$. We remind that two conjugate kernels define the same determinantal point process.

of the inner-product $\langle\!\langle\ ,\ \rangle\!\rangle$, as in (3.22). However, one easily verifies it with the above formulae. The equivalence of the Fredholm determinant parts is evident only after the change of variable $v \to 1/\tilde{u}$ and $u \to 1/\tilde{v}$. Then, the kernel obtained for $p_n^{(1)}$ is the transpose of the one for $p_n^{(2)}$.

Step 4: Expressing the kernel $\mathbb{K}_m(x,y)$ *as (3.12).* Using this new basis ψ_k, as in (3.21), and using the Christoffel-Darboux formula (3.25), the kernel $\mathbb{K}_m(x,y)$ becomes by Step 2 (recall that $n = 2m+1$):

$$
\begin{aligned}
\mathbb{K}_m(x,y) &= \sum_{k=1}^{n} \psi_k(x)\psi_k(y) \stackrel{*}{=} \oint_{S^1} d\rho_t(z) \oint_{\Gamma_{0,z}} d\rho_t(w) \frac{w^{y-m}}{z^{x-m}} \sum_{k=0}^{n-1} P_k(z^{-1})P_k(w) \\
&= \oint_{\Gamma_0} d\rho_t(z) \oint_{\Gamma_{0,z}} d\rho_t(w) \frac{w^{y-m}}{z^{x-m-1}} \frac{1}{z-w} \left(\left(\frac{w}{z}\right)^n P_n(z)P_n(w^{-1}) - P_n(z^{-1})P_n(w) \right),
\end{aligned}
\tag{3.38}
$$

Note that the w-integrand in the double integral $\stackrel{*}{=}$ has no pole at $w = z$, enabling one to deform the w-contour so as to include $z \in S^1$; this has the advantage that the double integral of the difference can be written as the difference of two double integrals, each of them being finite.

Inserting (3.36) into (3.38) and setting $V_m = 1/(H_{2m+1}(0)H_{2m+2}(0))$ we get

$$
\begin{aligned}
\mathbb{K}_m(x,y) = {} & \frac{V_m}{(2\pi \mathrm{i})^2} \oint_{\Gamma_0} dz \oint_{\Gamma_{0,z}} dw \frac{e^{t(z-z^{-1})}}{e^{t(w-w^{-1})}} \frac{w^{y-m-1}}{z^{x-m}} \frac{H_{2m+1}(w)H_{2m+1}(z^{-1})}{z-w} \\
& - \frac{V_m}{(2\pi \mathrm{i})^2} \oint_{\Gamma_0} dz \oint_{\Gamma_{0,z}} dw \frac{e^{t(w-w^{-1})}}{e^{t(z-z^{-1})}} \frac{w^{y+m}}{z^{x+m+1}} \frac{H_{2m+1}(z)H_{2m+1}(w^{-1})}{z-w}.
\end{aligned}
\tag{3.39}
$$

The expression in (3.12) is finally obtained by noticing that

$$
\frac{1}{(2\pi \mathrm{i})^2} \oint_{\Gamma_0} dz \oint_{\Gamma_{0,z}} dw \frac{F(z,w)}{w-z} = \frac{1}{(2\pi \mathrm{i})^2} \oint_{\Gamma_0} dw \oint_{\Gamma_{0,w}} dz \frac{F(z,w)}{w-z} + \oint_{\Gamma_0} \frac{dz}{2\pi \mathrm{i}} F(z,z),
\tag{3.40}
$$

proving formula (3.12).

Step 5: Expressing the kernel $\widetilde{\mathbb{K}}_m(x,y)$ *as (3.13).* First of all, by (3.36), we have

$$
H_n(z^{-1}) = P_n(z) e^{2t/z} z^{-n} \sqrt{H_n(0)H_{n+1}(0)}. \tag{3.41}
$$

Thus (with $n = 2m+1$), the last term of (3.12) is given by

$$
\frac{V_m}{2\pi \mathrm{i}} \oint_{\Gamma_0} \frac{dz}{z^{x-y+1}} H_{2m+1}(z^{-1})H_{2m+1}(z) = \frac{1}{2\pi \mathrm{i}} \oint_{S^1} \frac{dz}{z^{x-y+1}} e^{2t(z+z^{-1})} P_n(z)P_n(z^{-1}).
\tag{3.42}
$$

In particular, at $x = y$ we have

$$(3.42)\big|_{x=y} = \langle\!\langle P_n, P_n \rangle\!\rangle = 1, \tag{3.43}$$

and thus

$$\begin{aligned} &\frac{V_m}{2\pi \mathrm{i}} \oint_{\Gamma_0} \frac{dz}{z^{x-y+1}} H_{2m+1}(z^{-1}) H_{2m+1}(z) \\ &= \delta_{x,y} + (1-\delta_{x,y}) \frac{V_m}{2\pi \mathrm{i}} \oint_{\Gamma_0} dz \frac{1}{z^{x-y+1}} H_{2m+1}(z^{-1}) H_{2m+1}(z). \end{aligned} \tag{3.44}$$

So, $\widetilde{\mathbb{K}}_m(x,y) = \delta_{x,y} - \mathbb{K}_m(x,y) = \widetilde{\mathbb{K}}_m^{\mathrm{ext}}(0,x_1;0,x_2)$ of (2.5), thus establishing Theorem 3.1. This also ends the proof of Theorem 2.1 for $t_1 = t_2 = 0$. $\square$

4 Reshaping, motivation and Bessel representation

In this section we first reshape the kernel (2.5) of Theorem 2.1 for $t_1 = t_2 = 0$, to make it adequate for asymptotic analysis. Secondly, we rewrite all the terms using Bessel functions and the Bessel kernel. This will allow us to use known asymptotics for Bessel functions and kernel, without the need for new asymptotic analysis.

4.1 Reshaping

Note that the kernel $K(z^{-1})$, defined in (3.15), with $|u| < |v| < |z|$, namely

$$K(z^{-1})_{k,\ell} := \frac{(-1)^{k+\ell}}{(2\pi \mathrm{i})^2} \oint_{\Gamma_0} du \oint_{\Gamma_{0,u}} dv \frac{u^\ell}{v^{k+1}} \frac{1}{v-u} \frac{u-z}{v-z} \frac{e^{2t(u-u^{-1})}}{e^{2t(v-v^{-1})}}, \tag{4.1}$$

is a rank-one perturbation

$$K(z^{-1})_{k,\ell} = K(0)_{k,\ell} + h_k(z^{-1})\, g_\ell, \tag{4.2}$$

of the symmetric[7] kernel

$$K(0)_{k,\ell} = \frac{(-1)^{k+\ell}}{(2\pi \mathrm{i})^2} \oint_{\Gamma_0} du \oint_{\Gamma_{0,u}} dv \frac{u^\ell}{v^{k+1}} \frac{1}{v-u} \frac{e^{2t(u-u^{-1})}}{e^{2t(v-v^{-1})}}, \tag{4.3}$$

upon using the identity

$$\frac{1}{v-u}\frac{u-z}{v-z} = \frac{1}{v-u} - \frac{1}{v-z}, \tag{4.4}$$

[7] as is seen by replacing $u \mapsto 1/u,\ v \mapsto 1/v$.

where (remember $|v| < |z|$ in the first integration below)

$$\begin{aligned} h_k(z^{-1}) &= \frac{-1}{2\pi \mathrm{i}} \oint_{\Gamma_0} \frac{dv}{(-v)^{k+1}} \frac{e^{-2t(v-v^{-1})}}{v-z} \\ &= \frac{-1}{2\pi \mathrm{i}} \oint_{\Gamma_{0,z}} \frac{dv}{(-v)^{k+1}} \frac{e^{-2t(v-v^{-1})}}{v-z} + \frac{e^{-2t(z-z^{-1})}}{(-z)^{k+1}} \\ &=: \bar{h}_k(z^{-1}) + \frac{e^{-2t(z-z^{-1})}}{(-z)^{k+1}}, \end{aligned} \tag{4.5}$$

and

$$g_\ell = \frac{-1}{2\pi \mathrm{i}} \oint_{\Gamma_0} du\, (-u)^\ell e^{2t(u-u^{-1})}. \tag{4.6}$$

In (4.5), one has replaced the integration about a small circle around 0 by an integration about a contour containing z as well; this is done in order to be able to expand, later on, $1/(v-z)$ in a power series in z/v. Therefore we can rewrite the Fredholm determinant $H_n(z^{-1})$ of $K(z^{-1})$ as

$$H_n(z^{-1}) = H_n(0)(1 - R_n(z^{-1})), \tag{4.7}$$

where[8]

$$R_n(z^{-1}) := \langle Q, \chi_n h(z^{-1})\rangle, \quad Q_k := ((\mathbb{1} - \chi_n K(0)\chi_n)^{-1}\chi_n g)_k \tag{4.8}$$

and $\chi_n(k) = \mathbb{1}_{[k\geq n]}$; here the symmetry of $K(0)$ is being used. Accordingly $R_n(z^{-1}) = \langle Q, \chi_n h(z^{-1})\rangle$, as in (4.8), decomposes as (recall that $n = 2m+1$)

$$R_n(z^{-1}) = S_n(z^{-1}) + \frac{e^{-2t(z-z^{-1})}}{(-z)^n} T_n(z^{-1}), \tag{4.9}$$

with

$$S_n(z^{-1}) = \langle Q, \chi_n \bar{h}(z^{-1})\rangle, \quad T_n(z^{-1}) = \sum_{k\geq 1} \frac{Q_{n+k-1}}{(-z)^k}. \tag{4.10}$$

We set for $x \in \mathbb{Z}$,

$$\begin{aligned} A(x) &:= \frac{-1}{2\pi \mathrm{i}} \oint_{\Gamma_0} dz \frac{e^{t(z-z^{-1})}}{(-z)^{x-m}} (1 - S_n(z^{-1})), \\ B(x) &:= \frac{-1}{2\pi \mathrm{i}} \oint_{\Gamma_0} dz \frac{e^{-t(z-z^{-1})}}{(-z)^{x+m+1}} T_n(z^{-1}), \\ C_1(x) &:= \frac{-1}{2\pi \mathrm{i}} \oint_{\Gamma_0} dz \frac{T_n(z^{-1})T_n(z)}{(-z)^{x+1}}, \\ C_2(x) &:= \mathbb{1}_{[x\neq 0]} \frac{-1}{2\pi \mathrm{i}} \oint_{\Gamma_0} dz \frac{R_n(z^{-1}) + R_n(z) - R_n(z^{-1})R_n(z)}{(-z)^{x+1}} \\ C(x) &:= 2C_1(x) + C_2(x). \end{aligned} \tag{4.11}$$

[8]For $a = (a_k)_{k\in\mathbb{Z}}$ and $b = (b_k)_{k\in\mathbb{Z}}$, the inner-product $\langle a, b\rangle := \sum_{k\in\mathbb{Z}} a_k b_k$.

Remark that $C_1(x) = C_1(-x)$ and $C_2(x) = C_2(-x)$. Also introduce functions $E_i(z,w)$, which also depend on $n = 2m+1$,

$$\begin{aligned}
E_1(z,w) &:= \frac{e^{t(z-z^{-1})}}{e^{t(w-w^{-1})}} \left(\frac{z}{w}\right)^m (1-S_n(z^{-1}))(1-S_n(w)) \\
E_2(z,w) &:= -\frac{e^{t(z-z^{-1})}}{e^{-t(w-w^{-1})}} (-z)^m (-w)^{m+1} (1-S_n(z^{-1})) T_n(w) \\
E_3(z,w) &:= -\frac{e^{-t(z-z^{-1})}}{e^{t(w-w^{-1})}} (-z)^{-m-1} (-w)^{-m} T_n(z^{-1})(1-S_n(w)) \\
E_4(z,w) &:= -\frac{e^{t(z-z^{-1})}}{e^{t(w-w^{-1})}} \left(\frac{z}{w}\right)^m T_n(z) T_n(w^{-1}).
\end{aligned} \tag{4.12}$$

With these notations, the following statement hold.

Proposition 4.1. *The kernel $\widetilde{\mathbb{K}}_m(x,y)$ in (3.13) has the following expression*

$$\begin{aligned}
&(-1)^{x-y} \frac{H_{n+1}(0)}{H_n(0)} \widetilde{\mathbb{K}}_m(x,y) = C(x-y) \\
&\quad + \frac{1}{(2\pi \mathrm{i})^2} \oint_{\Gamma_0} dz \oint_{\Gamma_{0,z}} dw \, \frac{\sum_{i=1}^4 E_i(z,w)}{z-w} \left(\frac{(-w)^{y-1}}{(-z)^x} + \frac{(-z)^y}{(-w)^{x+1}} \right),
\end{aligned} \tag{4.13}$$

as well as the Airy kernel-like expression

$$\begin{aligned}
(-1)^{x-y} \frac{H_{n+1}(0)}{H_n(0)} \widetilde{\mathbb{K}}_m(x,y) &= C(x-y) \\
&+ \sum_{c \geq 0} \begin{pmatrix} A(x-c)A(y-c) + A(-x-c)A(-y-c) \\ -A(x-c)B(y-c) - A(-x-c)B(-y-c) \\ -B(x-c)A(y-c) - B(-x-c)A(-y-c) \end{pmatrix} \\
&- \sum_{c<0} \Big(B(x-c)B(y-c) + B(-x-c)B(-y-c) \Big).
\end{aligned} \tag{4.14}$$

Proof. Let us first prove (4.13). Consider the kernel $\widetilde{\mathbb{K}}_m(x,y)$ as in (3.13); one uses $H_n(z^{-1}) = H_n(0)(1-R_n(z^{-1}))$, as in (4.7), and one renames the integration variables $(w,z) \to (z,w)$ in the second double integral, enabling us to combine the two double integrals. Then, taking into account the prefactor,

$$\begin{aligned}
(-1)^{x-y} \frac{H_{n+1}(0)}{H_n(0)} \widetilde{\mathbb{K}}_m(x,y) &= \frac{\mathbb{1}_{[x \neq y]}}{2\pi \mathrm{i}} \oint_{\Gamma_0} \frac{dz}{(-z)^{x-y+1}} (1-R_n(z^{-1}))(1-R_n(z)) \\
+ \frac{1}{(2\pi \mathrm{i})^2} \oint_{\Gamma_0} dz \oint_{\Gamma_{0,z}} dw & \frac{e^{t(z-z^{-1})}}{e^{t(w-w^{-1})}} \left(\frac{z}{w}\right)^m \left(\frac{(-w)^{y-1}}{(-z)^x} + \frac{(-z)^y}{(-w)^{x+1}} \right) \\
&\times \frac{(1-R_n(z^{-1}))(1-R_n(w))}{z-w}.
\end{aligned} \tag{4.15}$$

That the single integral above equals C_2, defined in (4.11), follows from the fact that the -1 term can be deleted, since $\frac{1}{2\pi \mathrm{i}}\oint_{\Gamma_0} dz\, z^{y-x-1} = \delta_{x,y}$ and $\delta_{x,y}\mathbb{1}_{x\neq y} = 0$. Multiply out $(1-R_n(z^{-1}))(1-R_n(w))$, use the expression (4.10) of R_n and the functions E_i's defined in (4.12) with the result

$$
\begin{aligned}
(4.15) = & \frac{1}{(2\pi \mathrm{i})^2}\oint_{\Gamma_0} dz \oint_{\Gamma_{0,z}} dw \frac{1}{z-w}\left(\frac{(-w)^{y-1}}{(-z)^x} + \frac{(-z)^y}{(-w)^{x+1}}\right) \\
& \times \Big(E_1(z,w) + E_2(z,w) + E_3(z,w) - \frac{w}{z}E_4(w,z)\Big) + C_2(x-y).
\end{aligned} \tag{4.16}
$$

The double integral, involving the last expression in brackets, is not in a usable form, in view of the saddle point method and the topology of the contours (see the discussion after the proof). Namely, the *integrations have to be interchanged*, at the expense of a residue term, as is given by the general formula (3.40). So, using this formula, and further renaming $z \leftrightarrow w$, the double integral with E_4 becomes

$$
\frac{1}{(2\pi \mathrm{i})^2}\oint_{\Gamma_0} dz \oint_{\Gamma_{0,z}} dw \frac{1}{z-w}\left(\frac{(-w)^{y-1}}{(-z)^x} + \frac{(-z)^y}{(-w)^{x+1}}\right) E_4(z,w) + 2C_1(x-y), \tag{4.17}
$$

where $C_1(x)$ is defined in (4.11). So, taking equation (4.16) and (4.17) into account, we find that formula (4.13) for the kernel $\widetilde{\mathbb{K}}_m(x,y)$ holds.

Next we prove (4.14). The first observation is that the kernel (4.13) depends on x and y through the expression in brackets only; the latter itself is invariant for the interchange $(x,y) \mapsto (-y,-x)$. So it suffices to consider the double integral associated with the first term $(-w)^{y-1}(-z)^{-x}$ only; the other one is automatic. Since the integration paths can be taken to satisfy $|z| < |w|$, in the double integral of (4.13), one may use the series

$$
\frac{1}{z-w} = \frac{1}{(-w)}\sum_{c\geq 0}\left(\frac{-z}{-w}\right)^c, \quad \text{valid for } |w| > |z|, \tag{4.18}
$$

and one notices that for each of the E_i, the double integral decouples into the product of two integrals over Γ_0:

$$
\begin{aligned}
& \frac{1}{(2\pi \mathrm{i})^2}\oint_{\Gamma_0} dz \oint_{\Gamma_{0,z}} dw \frac{E_1(z,w)}{z-w}\frac{(-w)^{y-1}}{(-z)^x} \\
& = \sum_{c\geq 0}\oint_{\Gamma_0}\frac{-dz}{2\pi \mathrm{i}}\frac{e^{t(z-z^{-1})}}{(-z)^{x-m-c}}(1-S_n(z^{-1}))\oint_{\Gamma_0}\frac{-dw}{2\pi \mathrm{i}}\frac{(-w)^{y-m-c-2}}{e^{t(w-w^{-1})}}(1-S_n(w)) \\
& = \sum_{c\geq 0} A(x-c)A(y-c).
\end{aligned} \tag{4.19}
$$

To see that the second integral equals $A(y-c)$, one performs the change of variable $z \mapsto 1/z$. Since the only poles are at $z=0$ and $z^{-1}=0$, this is allowed; so, we do not pick up further poles. The same decoupling occurs for the other E_i's, which yields:

$$\begin{aligned}
\frac{1}{(2\pi \mathrm{i})^2}\oint_{\Gamma_0} dz \oint_{\Gamma_{0,z}} dw \frac{E_2(z,w)}{z-w}\frac{(-w)^{y-1}}{(-z)^x} &= -\sum_{c\geq 0} A(x-c)B(y-c) \\
\frac{1}{(2\pi \mathrm{i})^2}\oint_{\Gamma_0} dz \oint_{\Gamma_{0,z}} dw \frac{E_3(z,w)}{z-w}\frac{(-w)^{y-1}}{(-z)^x} &= -\sum_{c\geq 0} B(x-c)A(y-c)
\end{aligned} \tag{4.20}$$

and

$$\begin{aligned}
&\frac{1}{(2\pi \mathrm{i})^2}\oint_{\Gamma_0} dz \oint_{\Gamma_{0,z}} dw \frac{E_4(z,w)}{z-w}\frac{(-w)^{y-1}}{(-z)^x} \\
&= -\sum_{c\geq 0} B(-x+c+1)B(-y+c+1) = -\sum_{c<0} B(-x-c)B(-y-c).
\end{aligned} \tag{4.21}$$

Then adding the same expressions with the interchange $(x,y)\mapsto(-y,-x)$ yields formula (4.14), completing the proof of Proposition 4.1. $\square$

In anticipation of Section 7 on the integral representation of the limiting kernel, which will be obtained by saddle point analysis, some comments must be made here; they will also explain the interchange of integrals, which occurred in (4.17). Given the future rescaling $m \simeq 2t$ with $x=\xi_1 t^{1/3}$, $y=\xi_2 t^{1/3}$ for $t\to\infty$, the steep descent method applied to $A(x)$ and $B(x)$ at $z=-1$, in particular to the part of the integrand $e^{\pm t(z-z^{-1})}(-z)^{\pm m} = e^{\pm tF(z)}$ respectively, uses the Taylor expansions

$$\begin{aligned}
F(z) := z - z^{-1} + 2\log(-z) &= \frac{1}{3}(z+1)^3 + \mathcal{O}(z+1)^4, \\
\log(-z) = -(z+1) - \frac{1}{2}(z+1)^2 + \mathcal{O}(z+1)^3.
\end{aligned} \tag{4.22}$$

The steep descent path for $A(x)$ will therefore look like with an angle of approximately $\pm\pi/3$, whereas for $B(x)$ it will look like with an angle of approximately[9] $\pm 2\pi/3$ with the positive real axis. The contours of *the four double integrals* of equation (4.13), associated with each one of the E_i's, from the point of view of steepest descent analysis about $z,w=-1$, are topologically two circles, a z-circle inside a w-circle, which are deformed so that locally near $z=w=-1$ they look like the set of pictures in Figure 4 (see Section 7), with the two circles intersecting the real axis at the common point $z,w=-1$ and to the right of -1.

[9] The angles can be within the range $\pi/3\pm\pi/6$ and $2\pi/3\pm\pi/6$.

4.2 Bessel reformulation

The purpose of this section is to express the functions $A(x)$, $B(x)$, $C_1(x)$ and $C_2(x)$, as in (4.11) in terms of Bessel functions, the expressions Q_k and the Bessel kernel $K(0)$, as in (4.8) and (4.3). Throughout we will be using the integral representation of the Bessel function of order $n \in \mathbb{Z}$, together with its symmetries,

$$J_n(2t) = \frac{1}{2\pi \mathrm{i}} \oint_{\Gamma_0} dz \frac{e^{t(z-z^{-1})}}{z^{n+1}} = (-1)^n J_{-n}(2t) = (-1)^n J_n(-2t). \tag{4.23}$$

$J_n(2t)$ is different from the modified Bessel function $I_n(2t)$, defined in (3.4). To do so, we shall need the following Bessel function expressions for the basic building blocks.

Lemma 4.2. *The kernel $K(0)$ defined in (4.3), the expressions h_k and g_ℓ given in (4.5) and (4.6) and the functions $T_n(z^{-1})$ and $S_n(z^{-1})$, given in (4.10), can be expressed in terms of Bessel functions as follows:*

$$\begin{aligned} K(0)_{k,\ell} &= \sum_{a\geq 0} J_{\ell+a+1}(4t) J_{k+a+1}(4t) =: B_{2t}(k+1, \ell+1), \quad g_\ell = J_{\ell+1}(4t) \\ h_k(z^{-1}) &= -\sum_{a\geq 0} (-z)^a J_{k+a+1}(4t) + \frac{e^{-2t(z-z^{-1})}}{(-z)^{k+1}} = \bar{h}_k(z^{-1}) + \frac{e^{-2t(z-z^{-1})}}{(-z)^{k+1}}, \\ T_n(z^{-1}) &= \sum_{k\geq n} \frac{Q_k}{(-z)^{k-n+1}}, \quad S_n(z^{-1}) = -\sum_{\substack{a\geq 0 \\ k\geq n}} (-z)^a Q_k J_{k+a+1}(4t), \end{aligned} \tag{4.24}$$

where $B_t(i,j)$ is the Bessel kernel in [41]. Also,

$$Q_k = \sum_{\ell \geq n} P_{k,\ell} J_{\ell+1}(4t), \quad \textit{with } P_{k,\ell} = ((\mathbb{1} - \chi_n K(0) \chi_n)^{-1})_{k,\ell}. \tag{4.25}$$

Proof. For $K(0)_{k,\ell}$ one uses $1/(v-u) = v^{-1} \sum_{a\geq 0} (u/v)^a$ for $|u| < |v|$ and then (4.23). The same geometric series is used for $\bar{h}_k(z^{-1})$ in (4.5) but with u replaced by z, from which the formula (4.24) for $h_k(z^{-1})$ and the formula for S_n by (4.10) follow. Finally, one has $g_\ell = (-1)^{\ell-1} J_{-1-\ell}(2t) = J_{\ell+1}(2t)$. $\square$

The more intricate term is C_2 from (4.11).

Lemma 4.3. *The expression $C_2(x)$, as in (4.11), equals*

$$C_2(x) = \mathbb{1}_{[x\neq 0]} C_2^*(x), \tag{4.26}$$

where

$$\begin{aligned} C_2^*(x) &= (-1)^x \frac{1}{2\pi \mathrm{i}} \oint_{\Gamma_0} dz \frac{1}{z^{x+1}} \left(R_n(z^{-1}) + R_n(z) - R_n(z^{-1}) R_n(z) \right) \\ &= \sum_{k\geq n} Q_k \Big(\mathbb{1}_{[x\neq 0]} J_{k-|x|+1}(4t) - Q_{k+|x|} + \sum_{\ell\geq n} Q_\ell K(0)_{k,\ell-|x|} \Big). \end{aligned} \tag{4.27}$$

Proof. One first notices that the integrand in (4.27) is invariant under the mapping $z \mapsto z^{-1}$. Then, using the formula (4.24) for $R_n(z^{-1}) = \sum_{k\geq n} Q_k h_k(z^{-1})$, one breaks up the calculation as follows.
(a) Terms from $R_n(z^{-1}) + R_n(z)$. We have

$$R_n(z^{-1}) + R_n(z) = \sum_{k\geq n} Q_k (h_k(z^{-1}) + h_k(z)), \tag{4.28}$$

and thus, by integration, one checks first for $x > 0$, then for $x < 0$ and for $x = 0$, that, using the symmetry properties of the Bessel functions (see (4.23)),

$$\begin{aligned} \frac{1}{2\pi \mathrm{i}} \oint_{\Gamma_0} dz \frac{h_k(z^{-1}) + h_k(z)}{z^{x+1}} &= (-1)^x (\mathbb{1}_{x>0} J_{k+1-x}(4t) + \mathbb{1}_{x<0} J_{k+1+x}(4t)) \\ &= (-1)^x \mathbb{1}_{[x\neq 0]} J_{k+1-|x|}(4t). \end{aligned} \tag{4.29}$$

Substituting into the left hand side of (4.27) gives the first term on the right hand side of (4.27).
(b) Terms from $R_n(z^{-1}) R_n(z)$. We have

$$R_n(z^{-1}) R_n(z) = \sum_{k,\ell\geq n} Q_k Q_\ell h_k(z^{-1}) h_\ell(z). \tag{4.30}$$

From (4.23) and (4.24) we get

$$\begin{aligned} \frac{1}{2\pi \mathrm{i}} \oint_{\Gamma_0} dz \frac{h_k(z^{-1}) h_\ell(z)}{z^{x+1}} &= \sum_{a,b\geq 0} (-1)^{a-b} \delta_{a-b,x} J_{k+a+1}(4t) J_{b+\ell+1}(4t) \\ &\quad - \sum_{b\geq 0} (-1)^{b+k+1} J_{\ell+b+1}(4t) J_{k+b+1+x}(-4t) \\ &\quad - \sum_{a\geq 0} (-1)^{a+\ell+1} J_{k+a+1}(4t) J_{x-\ell-a-1}(4t) + (-1)^x \delta_{\ell-k,x} \\ &= (-1)^x \Big(\delta_{\ell-k,x} - \sum_{a\geq 0} J_{k+a+1}(4t) J_{\ell+a+1-|x|}(4t) \Big) \\ &= (-1)^x (\delta_{\ell-k,x} - K(0)_{k,\ell-|x|}), \end{aligned} \tag{4.31}$$

using in the last equality the expression (4.24) for the kernel $K(0)$. In the second equality we used the symmetries (4.23) of the Bessel functions. Substituted into (4.30), this gives the last two terms in (4.27). □

Proposition 4.4. *The expressions $A(x)$, $B(x)$, $C(x)$, defined in (4.11) for $x \in \mathbb{Z}$, can be expressed in terms of Bessel functions J_k, Q_k and the kernel $K(0)$, as follows:*

$$\begin{aligned} A(x) &= J_{m+1-x}(2t) + \sum_{k\geq n}\sum_{a\geq 0} Q_k J_{k+1+a}(4t) J_{m+1+a-x}(2t), \\ B(x) &= \sum_{k\geq n} Q_k J_{k-m+x}(2t), \end{aligned} \tag{4.32}$$

and

$$C(x) = \sum_{k\geq n} Q_k \left(J_{k-x+1}(4t) + J_{k+x+1}(4t)\right) + \sum_{k,\ell\geq n} Q_k Q_\ell \left(K(0)_{k+x,\ell} + K(0)_{k-x,\ell}\right). \tag{4.33}$$

Proof. The formulas for A and B follow directly from (4.11) and the expressions for T_n and S_n in (4.24), together with the symmetries (4.23) of the Bessel functions. Then

$$\begin{aligned} C_1(x) &= \frac{-1}{2\pi \mathrm{i}} \oint_{\Gamma_0} dz \frac{T_n(z^{-1})T_n(z)}{(-z)^{x+1}} \\ &= \sum_{k,\ell\geq n} Q_k Q_\ell \frac{(-1)^x}{2\pi \mathrm{i}} \oint_{\Gamma_0} dz \frac{(-z)^{k-\ell}}{z^{x+1}} = \sum_{k,\ell\geq n} Q_k Q_\ell \delta_{k-\ell,x} = \sum_{k\geq n} Q_k Q_{k+|x|}. \end{aligned} \tag{4.34}$$

From Lemma 4.3, it follows that

$$C_2(x) = \mathbb{1}_{[x\neq 0]} \sum_{k\geq n} Q_k \bigg(J_{k-|x|+1}(4t) - Q_{k+|x|} + \sum_{\ell\geq n} Q_\ell K(0)_{k,\ell-|x|} \bigg). \tag{4.35}$$

Next we show that $\mathbb{1}_{[x\neq 0]}$ can actually be omitted. To do so, it suffices to show that the sum on the right hand side of (4.35) vanishes when $x = 0$.

Indeed, setting $P = (\mathbb{1} - \chi_n K(0)\chi_n)^{-1}$, as in (4.25), remember that $g_\ell = J_{\ell+1}(4t)$ and that $Q_k = (P\chi_n g)_k$. Then, denoting $\langle \cdot, \cdot \rangle$ the canonical scalar product on $\ell^2(\mathbb{Z})$ we get, for $x = 0$, that the r.h.s. of (4.35) equals

$$\begin{aligned} &\langle P\chi_n g, \chi_n g\rangle - \langle P\chi_n g, \chi_n P\chi_n g\rangle + \langle P\chi_n g, \chi_n K(0)\chi_n P\chi_n g\rangle \\ &= \langle P\chi_n g, \chi_n g\rangle - \langle P\chi_n g, \chi_n(\mathbb{1} - \chi_n K(0)\chi_n) P\chi_n g\rangle \\ &= \langle P\chi_n g, \chi_n g\rangle - \langle P\chi_n g, \chi_n g\rangle = 0. \end{aligned} \tag{4.36}$$

Plugging there results into $C(x) = 2C_1(x) + C_2(x)$ we obtain

$$C(x) = \sum_{k \geq n} Q_k \left(J_{k-|x|+1}(4t) + Q_{k+|x|} + \sum_{\ell \geq n} Q_\ell K(0)_{k,\ell-|x|} \right). \tag{4.37}$$

It follows from the relation $P = \mathbb{1} + \chi_n K(0) \chi_n P$ (see the definition of Q and P in (4.25)) that acting on $\chi_n g$ and taking the kth entry,

$$Q_k = \mathbb{1}_{[k \geq n]} \left(J_{k+1}(4t) + \sum_{\ell \geq n} K(0)_{k,\ell} Q_\ell \right). \tag{4.38}$$

Using this relation for $Q_{k+|x|}$ in (4.37) we obtain

$$\begin{aligned} C(x) = &\sum_{k \geq n} Q_k \left(J_{k-|x|+1}(4t) + J_{k+|x|+1}(4t) \right) \\ &+ \sum_{k,\ell \geq n} Q_k Q_\ell \left(K(0)_{k,\ell-|x|} + K(0)_{k+|x|,\ell} \right). \end{aligned} \tag{4.39}$$

Finally, since $K(0)$ is symmetric, we replace $K(0)_{k,\ell-|x|} = K(0)_{\ell-|x|,k}$ and change the labeling $k \leftrightarrow \ell$ to get (4.33). Notice that we can then replace $|x|$ by x in this expression. □

5 Extended kernel for finite time

Formula (4.14) (in Proposition 4.1) with $A(x), B(x), C(x)$ given by Proposition 4.4 gives the kernel $\widetilde{\mathbb{K}}_m$ governing the fluctuations of the walkers near the point of meeting of the two groups of non-intersecting random walkers at time $\tau = 0$. In this section we prove Theorem 2.1 and we extend Proposition 4.1 to the multitime setting (Theorem 5.4).

Consider the $n = 2m+1$ walks whose positions were denoted by $x_k(\tau)$ in Section 3. Consider p different time slices $\tau_1 < \tau_2 < \ldots < \tau_p$ in the interval $(-t, t)$. Then, the probability measure at these times of the positions of the random walks is given by

$$\begin{aligned} &\mathbb{P}\left(\bigcap_{j=1}^{p} \bigcap_{k=1}^{n} \{x_k(\tau_j) = y_k^j\} \Big| \bigcap_{k=1}^{n} \{x_k(t) = x_k(-t) = m+1-k\} \right) \\ &= \text{const} \times \det \left[p_{t+\tau_1}(m+1-i, y_j^1) \right]_{1 \leq i,j \leq n} \\ &\times \left(\prod_{\ell=1}^{p-1} \det \left[p_{\tau_{\ell+1}-\tau_\ell}(y_i^\ell, y_j^{\ell+1}) \right]_{1 \leq i,j \leq n} \right) \det \left[p_{t-\tau_p}(y_i^p, m+1-j) \right]_{1 \leq i,j \leq n}. \end{aligned} \tag{5.1}$$

It is well known that a measure of this form has determinantal correlations in space-time [19, 22, 27, 37, 49], as stated in the following proposition.

Theorem 5.1. *Any probability measure on* $\{x_i^{(\ell)}, 1 \le i \le n, 1 \le \ell \le p\}$ *of the form*[10]

$$\frac{1}{Z}\det\left(\phi(\tau_0, a_i; \tau_1, x_j^{(1)})\right)_{1\le i,j\le n}\prod_{\ell=1}^{p-1}\det\left(\phi(\tau_\ell, x_i^{(\ell)}; \tau_{\ell+1}, x_j^{(\ell+1)})\right)_{1\le i,j\le n} \times \det\left(\phi(\tau_p, x_i^{(p)}; \tau_{p+1}, b_j)\right)_{1\le i,j\le n}, \tag{5.2}$$

has, assuming $Z \neq 0$, *the following determinantal k-point correlation functions for* $t_1, \dots, t_k \in \{\tau_1, \dots, \tau_p\}$:

$$\rho^{(k)}(t_1, x_1, \dots, t_k, x_k) = \det\left(K(t_i, x_i; t_j, x_j)\right)_{1\le i,j\le k}. \tag{5.3}$$

The space-time kernel K *(often called extended kernel) is given by*

$$\begin{aligned} K(t_1, x_1; t_2, x_2) = &-\phi(t_1, x_1; t_2, x_2)\mathbb{1}(t_2 > t_1) \\ &+ \sum_{i,j=1}^{n}\phi(t_1, x_1; \tau_{p+1}, b_i)[B^{-1}]_{i,j}\phi(\tau_0, a_j; t_2, x_2) \end{aligned} \tag{5.4}$$

with ($*$ *means integration with regard to the consecutive dots)*

$$\phi(\tau_r, x; \tau_s, y) = \begin{cases} \phi(\tau_r, x; \tau_{r+1}, \cdot) * \cdots * \phi(\tau_{s-1}, \cdot; \tau_s, y), & \text{if } \tau_r < \tau_s, \\ 0, & \text{if } \tau_r \ge \tau_s, \end{cases} \tag{5.5}$$

and with the $n \times n$ *matrix* B *having entries* $B_{i,j} = \phi(\tau_0, a_i; \tau_{p+1}, b_j)$.

Our measure (5.1) has the form required by Theorem 5.1. The normalization constant Z is nothing else but the partition function and it is non-zero since the set of n paths satisfying the non-intersection constraint is non-empty. We already determined the one-time kernel for $\tau = 0$. To get the extended kernel one has to let the one-time kernel"evolve" by means of the operator of the random walk. This formulation was already present in the work of Prähofer and Spohn on the Airy_2 process [41].

Lemma 5.2. *The extended kernel* $\widetilde{\mathbb{K}}_m^{\rm ext}$ *of the time-dependent point process* $\tilde\eta(\tau, x)$ *is given in terms of the kernel* $\widetilde{\mathbb{K}}_m$ *of the same point process at* $\tau = 0$ *by the formula:*

$$\widetilde{\mathbb{K}}_m^{\rm ext}(t_1, x_1; t_2, x_2) = -\mathbb{1}_{[t_2<t_1]}\left(e^{(t_2-t_1)\mathcal{H}}\right)(x_1, x_2) + \left(e^{-t_1\mathcal{H}}\widetilde{\mathbb{K}}_m e^{t_2\mathcal{H}}\right)(x_1, x_2), \tag{5.6}$$

[10] The functions $\phi(\tau_\ell, x; \tau_{\ell+1}, y)$ themselves may in fact vary with ℓ above.

where the infinitesimal generator $\mathcal{H}$ of the single random walk, the discrete Laplacian, acts on functions f as

$$\mathcal{H}f(x) = f(x+1) + f(x-1) - 2f(x), \quad x \in \mathbb{Z}. \tag{5.7}$$

Comparing the first term of (5.4) and (5.6), one sees a different ordering in the times. This is consequence of the dual transformation.

Proof of Lemma 5.2. The operator $\mathcal{H}$ in (5.7) is the generator of the continuous time process defined by the transition probability $p_t(x,y)$, in (2.1). Indeed, one checks that this transition probability is given by

$$p_t(x,y) = e^{-2t} I_{|x-y|}(2t) = \frac{1}{2\pi \mathrm{i}} \oint_{\Gamma_0} dz \, \frac{e^{t(z+z^{-1}-2)}}{z^{x-y+1}} = e^{t\mathcal{H}} \mathbb{1}(x,y) = (e^{t\mathcal{H}})(x,y), \tag{5.8}$$

because

$$\begin{aligned} \frac{\partial}{\partial t} p_t(x,y) &= \frac{1}{2\pi \mathrm{i}} \oint_{\Gamma_0} \frac{dz}{z^{x-y+1}} (z + z^{-1} - 2) e^{t(z+z^{-1}-2)} \\ &= p_t(x-1,y) + p_t(x+1,y) - 2p_t(x,y) = (\mathcal{H} p_t)(x,y) \end{aligned} \tag{5.9}$$

with initial conditions $p_0(x,y) = \mathbb{1}(x,y)$. Here, $\mathbb{1}$ denotes the identity operator on $\mathbb{Z}$, i.e., $\mathbb{1}(x,y) = 1$ if $x = y$ and $\mathbb{1}(x,y) = 0$ if $x \neq y$. The one-point kernel in Section 3, formula (3.38), was written as a sum involving $\psi_k(x)$ and $\psi_k(y)$. Under the time flow, they will become different functions; therefore, we set $\Psi_k(0,x) = \Phi_k(0,x) = \psi_k(x)$, and thus, with this new notation, the kernel reads

$$\mathbb{K}_m(x_1,x_2) = \sum_{k=1}^{n} \psi_k(x_1)\psi_k(x_2) = \sum_{k=1}^{n} \Psi_k(0,x_1)\Phi_k(0,x_2). \tag{5.10}$$

The two set of functions $\{\Phi_k(0,x), k = 1,\dots,n\}$ and $\{\Psi_k(0,x), k = 1,\dots,n\}$ satisfy

$$\begin{aligned} \operatorname{span}\{\Phi_k(0,x), k = 1,\dots,n\} &= \operatorname{span}\{p_t(m+1-k,x), k = 1,\dots,n\}, \\ \operatorname{span}\{\Psi_k(0,x), k = 1,\dots,n\} &= \operatorname{span}\{p_t(x,m+1-k), k = 1,\dots,n\}, \\ \text{with} \quad \langle \Phi_k(0,x), &\Psi_p(0,x) \rangle = \delta_{k,p}, \end{aligned} \tag{5.11}$$

so that the matrix B defined in (5.4) becomes the identity matrix.

Let us consider the functions of Theorem 5.1. First of all, the function $\phi(t_1,x_1;t_2,x_2)$ appearing in (5.4) becomes

$$\phi(t_1,x_1;t_2,x_2)\mathbb{1}_{[t_2>t_1]} = \mathbb{1}_{[t_2>t_1]} p_{t_2-t_1}(x_1,x_2) = \mathbb{1}_{[t_2>t_1]} (e^{(t_2-t_1)\mathcal{H}})(x_1,x_2), \tag{5.12}$$

where $t_1, t_2 \in \{\tau_1, \ldots, \tau_p\}$. Next, with $\tau_0 = -t$, $\tau_{p+1} = t$ we have

$$\phi(t_1, x; t, b_k) = p_{t-t_1}(x, b_k) = (e^{-t_1\mathcal{H}})(x, \cdot) * \phi(0, \cdot\,; t, b_k) \tag{5.13}$$

and

$$\phi(-t, a_k; t_2, x) = p_{t_2+t}(a_k, x) = \phi(-t, a_k; 0, \cdot) * (e^{t_2\mathcal{H}})(\cdot, x). \tag{5.14}$$

With the choice of basis used for the kernel at $\tau = 0$, we have that $\phi(0, \cdot\,; t, b_k)$ is replaced by $\Psi_k(0, \cdot)$ and $\phi(-t, a_k; 0, \cdot)$ by $\Phi_k(0, \cdot)$ (so that $B = \mathbb{1}$). Thus in Theorem 5.1 we have replaced

$$\phi(t_1, x; t, b_k) \to (e^{-t_1\mathcal{H}})(x, \cdot) * \Psi_k(0, \cdot) = (e^{-t_1\mathcal{H}}\Psi_k(0, \cdot))(x) =: \Psi_k(t_1, x) \tag{5.15}$$

and

$$\begin{aligned}\phi(-t, a_k; t_2, x) \to \Phi_k(0, \cdot) * (e^{t_2\mathcal{H}})(\cdot, x) &= (\Phi_k(0, \cdot)e^{t_2\mathcal{H}})(x) \\ &= (e^{t_2\mathcal{H}^\top}\Phi_k(0, \cdot))(x) =: \Phi_k(t_2, x).\end{aligned} \tag{5.16}$$

Therefore the extended kernel has the following expression in terms of the kernel $\mathbb{K}_m$ in (3.38)

$$\begin{aligned}\mathbb{K}_m^{\text{ext}}(t_1, x_1; t_2, x_2) &= -\mathbb{1}_{[t_1<t_2]}p_{t_2-t_1}(x_1, x_2) + \sum_{k=1}^{n}\Psi_k(t_1, x_1)\Phi_k(t_2, x_2) \\ &= -\mathbb{1}_{[t_1<t_2]}(e^{(t_2-t_1)\mathcal{H}})(x_1, x_2) + \left(e^{-t_1\mathcal{H}}\mathbb{K}_m e^{t_2\mathcal{H}}\right)(x_1, x_2).\end{aligned} \tag{5.17}$$

Notice that, using the semi-group property of $e^{t\mathcal{H}}$, we have the consistency relations (for $i = 1, \ldots, p$)

$$\begin{aligned}\Psi_k(\tau_i, x) &= \left(e^{(\tau_p-\tau_i)\mathcal{H}}\Psi_k(\tau_p, \cdot)\right)(x), \\ \Phi_k(\tau_i, x) &= \left(\Phi_k(\tau_1, \cdot)e^{(\tau_i-\tau_1)\mathcal{H}}\right)(x).\end{aligned} \tag{5.18}$$

The kernel $\widetilde{\mathbb{K}}_m^{\text{ext}}$ for the dual random walk is then given by taking the complement. Using (5.17) and remembering that $\widetilde{\mathbb{K}}_m = \mathbb{1} - \mathbb{K}_m$ from formula (3.10) we get

$$\begin{aligned}\widetilde{\mathbb{K}}_m^{\text{ext}}&(t_1, x_1; t_2, x_2) = \mathbb{1}_{[t_1=t_2]}\mathbb{1}(x_1, x_2) - \mathbb{K}_m^{\text{ext}}(t_1, x_1; t_2, x_2) \\ &= \mathbb{1}_{[t_1=t_2]}\mathbb{1}(x_1, x_2) + \mathbb{1}_{[t_1<t_2]}(e^{(t_2-t_1)\mathcal{H}})(x_1, x_2) - \left(e^{-t_1\mathcal{H}}\mathbb{K}_m e^{t_2\mathcal{H}}\right)(x_1, x_2) \\ &= \mathbb{1}_{[t_1=t_2]}(e^{(t_2-t_1)\mathcal{H}})(x_1, x_2) + \mathbb{1}_{[t_1<t_2]}(e^{(t_2-t_1)\mathcal{H}})(x_1, x_2) \\ &\quad - (e^{(t_2-t_1)\mathcal{H}})(x_1, x_2) + \left(e^{-t_1\mathcal{H}}(\mathbb{1} - \mathbb{K}_m)e^{t_2\mathcal{H}}\right)(x_1, x_2) \\ &= -\mathbb{1}_{[t_2<t_1]}(e^{(t_2-t_1)\mathcal{H}})(x_1, x_2) + (e^{-t_1\mathcal{H}}\widetilde{\mathbb{K}}_m e^{t_2\mathcal{H}})(x_1, x_2),\end{aligned} \tag{5.19}$$

yielding (5.6), ending the proof of Lemma 5.2. □

With the help of Lemma 5.2 we can easily prove Theorem 2.1 starting from Theorem 3.1.

Proof of Theorem 2.1. One of the key ingredients is that $f(x) := u^x$ is an eigenfunction of $\mathcal{H}$ with eigenvalue $u + u^{-1} - 2$. Indeed,

$$(\mathcal{H}f)(x) = u^{x+1} + u^{x-1} - 2u^x = (u + u^{-1} - 2)u^x = (u + u^{-1} - 2)f(x). \tag{5.20}$$

Moreover, $\mathcal{H}$ is symmetric. Therefore,

$$(e^{t\mathcal{H}} f)(x) = e^{t(u+u^{-1}-2)} f(x), \quad (f e^{t\mathcal{H}})(x) = (e^{t\mathcal{H}^\top} f)(x) = e^{t(u+u^{-1}-2)} f(x). \tag{5.21}$$

Then, (2.5) follows straightforwardly from (3.13) by applying $e^{-t_1\mathcal{H}}$ to the left, $e^{t_2\mathcal{H}}$ to the right of $\widetilde{\mathbb{K}}_m$ (together with (5.8) for the first term of (5.6)). □

For the further analysis, we extend the reformulation of the kernel for $\tau = 0$, as in Proposition 4.1, to the extended case. For that purpose, we first define the basic functions replacing A, B, and C of the one-time case (see Proposition 4.4). To do so, define a new function $J_x^{(\tau)}(2t)$ dependent on a parameter τ

$$J_x^{(\tau)}(2t) := \oint_{\Gamma_0} \frac{dz}{2\pi \mathrm{i} z} \frac{e^{t(z-z^{-1})}}{z^x} e^{\tau(z+z^{-1}-2)} = e^{-2\tau} \left(\frac{t+\tau}{t-\tau}\right)^{x/2} J_x(2\sqrt{t^2-\tau^2}). \tag{5.22}$$

Also define a τ-dependent extension of the kernel $K(0)_{k,\ell}$, as in (4.24), namely

$$K^{(\tau)}(0)_{k,\ell} := \sum_{a\geq 0} J_{a+k+1}^{(\tau)}(4t) J_{a+\ell+1}(4t). \tag{5.23}$$

Then define new functions $A(\tau,x), B(\tau,x), C(\tau,x)$ with $\tau \in \mathbb{R}$ and $x \in \mathbb{Z}$, which extend the functions $A(x), B(x), C(x)$, first defined in (4.11) and re-expressed in (4.32), by

$$\begin{aligned}
A(\tau,x) &:= J_{m+1-x}^{(\tau)}(2t) + \sum_{k\geq n}\sum_{a\geq 0} Q_k J_{k+1+a}(4t) J_{m+1+a-x}^{(\tau)}(2t), \\
B(\tau,x) &:= \sum_{k\geq n} Q_k J_{k-m+x}^{(\tau)}(2t), \\
C(\tau,x) &:= \sum_{k\geq n} Q_k \left(J_{k+1+x}^{(\tau)}(4t) + J_{k+1-x}^{(\tau)}(4t)\right) \\
&\quad + \sum_{k,\ell\geq n} Q_k Q_\ell \left(K^{(\tau)}(0)_{k+x,\ell} + K^{(\tau)}(0)_{k-x,\ell}\right).
\end{aligned} \tag{5.24}$$

Remember $H_n(0) = \det(\mathbb{1} - K(0))_{\ell^2(n,n+1,\dots)}$.

Lemma 5.3. *Given the notation (4.12) for the E_i's, the extended kernel $\widetilde{\mathbb{K}}_m^{\mathrm{ext}}$ is given by*

$$
\begin{aligned}
&\frac{(-1)^{x_2}e^{4t_2}}{(-1)^{x_1}e^{4t_1}}\frac{H_{n+1}(0)}{H_n(0)}\,\widetilde{\mathbb{K}}_m^{\mathrm{ext}}(t_1,x_1;t_2,x_2)\\
&\qquad\qquad = -\,\mathbb{1}_{[t_2<t_1]}p_{t_1-t_2}(x_1,x_2)\frac{H_{n+1}(0)}{H_n(0)}+C(t_1-t_2,x_1-x_2)\\
&\qquad\qquad\quad +\frac{1}{(2\pi\mathrm{i})^2}\oint_{\Gamma_0}dz\oint_{\Gamma_{0,z}}dw\frac{1}{z-w}\sum_{i=1}^{4}E_i(z,w)\\
&\qquad\qquad\quad \times\left(\frac{(-w)^{x_2-1}}{(-z)^{x_1}}\frac{e^{-t_1(z+z^{-1}+2)}}{e^{-t_2(w+w^{-1}+2)}}+\frac{(-z)^{x_2}}{(-w)^{x_1+1}}\frac{e^{-t_1(w+w^{-1}+2)}}{e^{-t_2(z+z^{-1}+2)}}\right).
\end{aligned}
\tag{5.25}
$$

Theorem 5.4. *The extended kernel $\widetilde{\mathbb{K}}_m^{\mathrm{ext}}$ is also expressed as*

$$
\begin{aligned}
&\frac{(-1)^{x_2}e^{4t_2}}{(-1)^{x_1}e^{4t_1}}\frac{H_{n+1}(0)}{H_n(0)}\,\widetilde{\mathbb{K}}_m^{\mathrm{ext}}(t_1,x_1;t_2,x_2)\\
&= -\mathbb{1}_{[t_2<t_1]}p_{t_1-t_2}(x_1,x_2)\frac{H_{n+1}(0)}{H_n(0)}+C(t_1-t_2,x_1-x_2)\\
&\quad+\sum_{c\geq 0}\Big(A(t_1,x_1-c)A(-t_2,x_2-c)+A(t_1,-x_1-c)A(-t_2,-x_2-c)\\
&\qquad\qquad -A(t_1,x_1-c)B(-t_2,x_2-c)-A(t_1,-x_1-c)B(-t_2,-x_2-c)\\
&\qquad\qquad -B(t_1,x_1-c)A(-t_2,x_2-c)-B(t_1,-x_1-c)A(-t_2,-x_2-c)\Big)\\
&\quad-\sum_{c<0}\Big(B(t_1,x_1-c)B(-t_2,x_2-c)+B(t_1,-x_1-c)B(-t_2,-x_2-c)\Big).
\end{aligned}
\tag{5.26}
$$

Proof of Lemma 5.3 and Theorem 5.4. First of all, let us focus on the term $\left(e^{(t_2-t_1)\mathcal{H}}\right)(x_1,x_2)$ in (5.6). Remember that $t_2-t_1<0$, so we can rewrite

$$
\begin{aligned}
\frac{(-1)^{x_2}e^{4t_2}}{(-1)^{x_1}e^{4t_1}}\left(e^{(t_2-t_1)\mathcal{H}}\right)(x_1,x_2) &= \frac{(-1)^{x_2}e^{2t_2}}{(-1)^{x_1}e^{2t_1}}I_{|x_1-x_2|}(2(t_2-t_1))\\
&= \frac{e^{2t_2}}{e^{2t_1}}I_{|x_1-x_2|}(2(t_1-t_2)) = p_{t_1-t_2}(x_1,x_2),
\end{aligned}
\tag{5.27}
$$

where we used the property $I_n(-2t)=(-1)^nI_n(2t)$ of the modified Bessel function; see (3.4).

Next we derive the double integrals in (5.25). The corresponding expression of the kernel $\widetilde{\mathbb{K}}_m$ in (4.13) is a linear combination (not forgetting the

conjugation factor of the l.h.s. of (4.13)) of

$$-\frac{w^{x_2-1}}{z^{x_1}}-\frac{z^{x_2}}{w^{x_1+1}}. \tag{5.28}$$

Applying $e^{-t_1\mathcal{H}}$ to the left and $e^{t_2\mathcal{H}}$ to the right, (5.28) transforms into

$$-\frac{w^{x_2-1}}{z^{x_1}}\frac{e^{-t_1(z+z^{-1}-2)}}{e^{-t_2(w+w^{-1}-2)}}-\frac{z^{x_2}}{w^{x_1+1}}\frac{e^{-t_1(w+w^{-1}-2)}}{e^{-t_2(z+z^{-1}-2)}}. \tag{5.29}$$

The multiplication by the prefactor $\frac{(-1)^{x_2}e^{4t_2}}{(-1)^{x_1}e^{4t_1}}$ leads then to the expression in (5.25).

Next derive the terms with the sums in (5.26) and the expression for C. We act with the semigroup on the summation part of the kernel (4.14), which is expressed in terms of $A(x),B(x),C(x)$, namely

$$\begin{aligned}\frac{H_{n+1}(0)}{H_n(0)}\widetilde{\mathbb{K}}_m(x_1,x_2)=&\sum_{c\geq 0}[(-1)^{x_1}A(x_1-c)][(-1)^{x_2}A(x_2-c)]+\dots\\&+(-1)^{x_1-x_2}C(x_1-x_2),\end{aligned} \tag{5.30}$$

with $A(x),B(x),C(x)$ given in Proposition 4.4. So, except for the term $C(x_1-x_2)$, the expression above is a sum of decoupled terms. Therefore acting on the $(-1)^xA(\pm x-c)$'s and $(-1)^xB(\pm x-c)$'s with $e^{-t_1\mathcal{H}}$ to the left amounts (by linearity) to acting on the $(-1)^xJ_{N\pm x}(2t)$ (for some N depending on the terms) and finally to acting on $1/(-z)^{\pm x}$ inside the integration. More precisely, by (5.21) with $f(x):=1/(-z)^{\pm x}$, we have

$$(e^{-t_1\mathcal{H}}f)(x)=e^{t_1(z+z^{-1}+2)}f(x),\quad\text{and}\quad(fe^{t_2\mathcal{H}})(x)=e^{-t_2(z+z^{-1}+2)}f(x), \tag{5.31}$$

from which, by linearity,

$$\begin{aligned}\sum_{y\in\mathbb{Z}}(e^{-t_1\mathcal{H}})(x,y)(-1)^yJ_{N\pm y}(2t)&=\oint_{\Gamma_0}\frac{dz}{2\pi\mathrm{i}z}\frac{e^{t(z-z^{-1})}}{z^N}(e^{-t_1\mathcal{H}}f)(x)\\&=\oint_{\Gamma_0}\frac{dz}{2\pi\mathrm{i}z}\frac{e^{t(z-z^{-1})}}{z^N(-z)^{\pm x}}e^{t_1(z+z^{-1}+2)}=(-1)^xe^{4t_1}J^{(t_1)}_{N\pm x}(2t),\end{aligned} \tag{5.32}$$

and

$$\sum_{y\in\mathbb{Z}}(-1)^yJ_{N\pm y}(2t)(e^{t_2\mathcal{H}})(y,x)=(-1)^xe^{-4t_2}J^{(-t_2)}_{N\pm x}(2t). \tag{5.33}$$

This extends to the functions $(-1)^xA(\pm x-c)$, $(-1)^xB(\pm x-c)$ because they are linear in the $(-1)^xJ_{N\pm x}(2t)$ (see (4.32)). Explicitly, applying $e^{-t_1\mathcal{H}}$ (to the

left) to $(-1)^x A(\pm x - c)$ amounts to replacing $A(\pm x - c)$ with $e^{4t_1} A(t_1, \pm x - c)$. Similarly, applying $e^{t_2 \mathcal{H}}$ (to the right) to $(-1)^x A(\pm x - c)$ amounts to replacing $A(\pm x - c)$ with $e^{-4t_2} A(-t_2, \pm x - c)$. The same holds for B instead of A. Thus we have obtained the terms in kernel (5.26) including A's and B's.

Exactly the same procedure applies for the term $(-1)^{x_1 - x_2} C(x_1 - x_2)$, because it is again a linear combination of $(-1)^{x_1 - x_2} J_{N \pm x_1 \mp x_2}(4t)$. Therefore acting with $e^{-t_1 \mathcal{H}}$ and $e^{t_2 \mathcal{H}}$ as before on $(-1)^{x_1 - x_2} C(x_1 - x_2)$ leads to the replacement of $C(x_1 - x_2)$ by $e^{4(t_1 - t_2)} C(t_1 - t_2, x_1 - x_2)$. This ends the proof of the formulas (5.25) and (5.26) for the extended kernel, thus establishing Lemma 5.3 and Theorem 5.4. $\square$

6 Asymptotics

In this section we prove the first half of Theorem 2.2, namely formula (2.20). From the discussion in Section 2 after Theorem 2.1, concerning the interaction between the top and bottom sets of random walks, we rescale space, time and the gap $n = 2m + 1$ between the two groups of walkers, as follows:

$$m = 2t + \sigma t^{1/3}, \quad x_i = \xi_i t^{1/3}, \quad t_i = s_i t^{2/3}, \quad i = 1, 2, \tag{6.1}$$

where $\sigma \in \mathbb{R}$ is a fixed parameter modulating the "strength of interaction" between the upper and lower sets of walks. To prove formula (2.20) of Theorem 2.2, we first analyze the asymptotics of the building blocks and determine some bounds which will be used later to show that we can exchange (by dominated convergence) the large time limit with the integrals (sums).

Recall from (5.22), (2.13) and (5.23) the functions $J_x^{(\tau)}(2t)$ and $\mathcal{Q}$, and the kernel $K^{(\tau)}(0)_{k,\ell}$:

$$\begin{aligned} J_x^{(\tau)}(2t) &= e^{-2\tau} \left(\frac{t+\tau}{t-\tau} \right)^{x/2} J_x(2\sqrt{t^2 - \tau^2}). \\ K^{(\tau)}(0)_{k,\ell} &= \sum_{a \geq 0} J^{(\tau)}_{a+k+1}(4t) J_{a+\ell+1}(4t). \\ \mathcal{Q}(\kappa) &= [(\mathbb{1} - \chi_{\tilde{\sigma}} K_{\mathrm{Ai}} \chi_{\tilde{\sigma}})^{-1} \chi_{\tilde{\sigma}} \mathrm{Ai}](\kappa), \quad \text{with } \tilde{\sigma} := 2^{2/3}\sigma, \end{aligned} \tag{6.2}$$

and where $\chi_a(x) = \mathbb{1}_{[x>a]}$. Remember from (2.14) the definition of

$$\mathrm{Ai}^{(s)}(\xi) := e^{\xi s + \frac{2}{3}s^3} \mathrm{Ai}(\xi + s^2), \tag{6.3}$$

and define the Airy-like kernel

$$K_{\mathrm{Ai}}^{(s)}(\kappa, \lambda) := \int_0^\infty d\gamma \, \mathrm{Ai}^{(s 2^{-2/3})}(\kappa + \gamma) \mathrm{Ai}(\lambda + \gamma). \tag{6.4}$$

Also define the following step functions of $\kappa, \lambda \in \mathbb{R}$, for which -by anticipation- we indicate the limits for $t \to \infty$:

$$\begin{aligned}
\mathcal{J}_t^{(s)}(\kappa) &:= t^{1/3} J^{(st^{2/3})}_{[2t+\kappa t^{1/3}+1]}(2t) \quad \longrightarrow \mathrm{Ai}^{(s)}(\kappa) \\
\mathcal{K}_t^{(s)}(\kappa,\lambda) &:= (2t)^{1/3} K^{(st^{2/3})}(0)_{[4t+\kappa(2t)^{1/3}],[4t+\lambda(2t)^{1/3}]} \quad \longrightarrow K^{(s)}_{\mathrm{Ai}}(\kappa,\lambda) \\
\mathcal{Q}_t(\kappa) &:= (2t)^{1/3} Q_{[4t+\kappa(2t)^{1/3}]} \\
&= \left[\left(\mathbb{1} - \chi_{\frac{n-4t}{(2t)^{1/3}}} \mathcal{K}_t^{(0)} \chi_{\frac{n-4t}{(2t)^{1/3}}}\right)^{-1} \chi_{\frac{n-4t}{(2t)^{1/3}}} \mathcal{J}_{2t}^{(0)}\right](\kappa) \quad \longrightarrow \mathcal{Q}(\kappa).
\end{aligned} \tag{6.5}$$

Lemma 6.1. *We have the following bounds and limits for* $\mathcal{J}_t^{(s)}$ *and* $\mathcal{K}_t^{(s)}$ *defined in (6.5). There exists a* $t_0 > 0$ *such that uniformly for* $t \geq t_0$ *it holds that*

$$|\mathcal{J}_t^{(s)}(\kappa)| \leq c_1 \min\{1, e^{-\theta\kappa}\}, \quad |\mathcal{K}_t^{(s)}(\kappa,\lambda)| \leq c_2 e^{-\theta(\kappa+\lambda)} \tag{6.6}$$

for any fixed $\theta > 0$ *and some constants* $c_1, c_2 > 0$ *(independent of* t*). Moreover*

$$\lim_{t\to\infty} \mathcal{J}_t^{(s)}(\kappa) = \mathrm{Ai}^{(s)}(\kappa), \quad \lim_{t\to\infty} \mathcal{K}_t^{(s)}(\kappa,\lambda) = K^{(s)}_{\mathrm{Ai}}(\kappa,\lambda) \tag{6.7}$$

uniformly for κ, λ *and* s *in a bounded set.*

Proof. We have

$$\begin{aligned}
\mathcal{J}_t^{(s)}(\xi) &= t^{1/3} J^{(st^{2/3})}_{[2t+\xi t^{1/3}]}(2t) \\
&= e^{-2st^{2/3}} \left(\frac{1+st^{-1/3}}{1-st^{-1/3}}\right)^{\frac{1}{2}[2t+\xi t^{1/3}]} t^{1/3} J_{[2t+\xi t^{1/3}]}\left(2t\sqrt{1-s^2t^{-2/3}}\right).
\end{aligned} \tag{6.8}$$

The prefactor can be estimated for $t \to \infty$, as follows

$$e^{-2st^{2/3}} \left(\frac{1+st^{-1/3}}{1-st^{-1/3}}\right)^{t+\frac{1}{2}\xi t^{1/3}} = e^{\xi s + \frac{2}{3}s^3}\left(1 + \mathcal{O}(t^{-1/3})\right) \tag{6.9}$$

where the $\mathcal{O}(t^{-1/3})$ is uniform for s in a bounded set and independent of ξ. Therefore, for t large enough, $|(6.9)| \leq \exp(2|\xi s| + |s^3|)$. Concerning the remaining part of (6.8), using (A.4) one readily obtains

$$\lim_{t\to\infty} t^{1/3} J_{[2t+\xi t^{1/3}]}\left(2t\sqrt{1-s^2t^{-2/3}}\right) = \mathrm{Ai}(\xi + s^2). \tag{6.10}$$

Regarding the bound, for s in a bounded set, if t is large enough it follows from the bound (A.6) that

$$\left| t^{1/3} J_{[2t+\xi t^{1/3}]}\left(2t\sqrt{1-s^2t^{-2/3}}\right)\right| \tag{6.11}$$

is first of all uniformly bounded and for large ξ it decays as $e^{-\beta\xi}$ for any choice of $\beta > 0$. The statements in the first parts of (6.6) and (6.7) then follow if we choose β satisfying $\beta \geq \theta + 2|s|$ for any s in the given bounded set.

To compute the limit of $\mathcal{K}_t^{(s)}$, one uses definition (6.5) and formula (6.2) for $K^{(st^{2/3})}(0)$, but with J replaced by $\mathcal{J}$ in the last equality below,

$$\begin{aligned}
\mathcal{K}_t^{(s)}(\kappa,\lambda) &= (2t)^{1/3} K^{(st^{2/3})}(0)_{[4t+\kappa(2t)^{1/3}],[4t+\lambda(2t)^{1/3}]} \\
&= (2t)^{1/3} \sum_{\gamma\in(2t)^{-1/3}\mathbb{N}} J^{(s2^{-2/3}(2t)^{2/3})}_{[4t+(\gamma+\kappa)(2t)^{1/3}]}(4t) J_{[4t+(\gamma+\lambda)(2t)^{1/3}]}(4t) \\
&= \frac{1}{(2t)^{1/3}} \sum_{\gamma\in(2t)^{-1/3}\mathbb{N}} \mathcal{J}_{2t}^{(s2^{-2/3})}(\kappa+\gamma)\mathcal{J}_{2t}^{(0)}(\lambda+\gamma).
\end{aligned} \tag{6.12}$$

From this, using the bound (6.6) on $\mathcal{J}$, we obtain

$$|\mathcal{K}_t^{(s)}(\kappa,\lambda)| \leq c_1^2 e^{-\theta(\kappa+\lambda)} \frac{1}{(2t)^{1/3}} \sum_{\gamma\in(2t)^{-1/3}\mathbb{N}} e^{-2\theta\gamma} \leq c_2 e^{-\theta(\kappa+\lambda)} \tag{6.13}$$

for $t \geq t_0 = 1$ and some $c_2 > 0$, uniformly for s in a bounded set.

We can think of the sum in (6.12) as an integral of piece-wise constant functions. The first bound in (6.6) allows us to use dominated convergence to exchange the limit and the integral. Then, $\lim_{t\to\infty} \mathcal{J}_t^{(s)}(\kappa) = \mathrm{Ai}^{(s)}(\kappa)$ yields

$$\lim_{t\to\infty} \mathcal{K}_t^{(s)}(\kappa,\lambda) = \int_0^\infty d\gamma\, \mathrm{Ai}^{(2^{-2/3}s)}(\kappa+\gamma)\mathrm{Ai}(\lambda+\gamma) = K_{\mathrm{Ai}}^{(s)}(\kappa,\lambda). \tag{6.14}$$

$\square$

Lemma 6.2. *Set $\tilde\sigma_t := \frac{n-4t}{(2t)^{1/3}}$ and define the operator $\mathcal{M}_t = \chi_{\tilde\sigma_t}\mathcal{K}_t^{(0)}\chi_{\tilde\sigma_t}$, appearing in the definition (6.5) of $\mathcal{Q}_t$. Then, uniformly for $t \geq t_0$, we have for the operator-norm*[11] *$\|\cdot\|$,*

$$\|\mathcal{M}_t\| < 1 \tag{6.15}$$

which implies that

$$\|(\mathbb{1} - \mathcal{M}_t)^{-1}\| \leq (1 - \|\mathcal{M}_t\|)^{-1} \leq C < \infty \tag{6.16}$$

for some finite constant C independent of t.

[11] where $\|A\| = \sup_{|f|\leq 1} |Af|$.

Proof. By Lemma 6.1 and the fact that $\tilde{\sigma}_t \to \tilde{\sigma}$ as $t \to \infty$, it follows that

$$\lim_{t\to\infty} \mathcal{M}_t = \chi_{\tilde{\sigma}} K_{\mathrm{Ai}} \chi_{\tilde{\sigma}} =: \mathcal{M} \tag{6.17}$$

pointwise. Moreover,

$$\begin{aligned} \lim_{t\to\infty} \|\mathcal{M}_t - \mathcal{M}\|^2 &\leq \lim_{t\to\infty} \|\mathcal{M}_t - \mathcal{M}\|_{\mathrm{HS}}^2 = \lim_{t\to\infty} \int d\kappa d\lambda \, |\mathcal{M}_t(\kappa,\lambda) - \mathcal{M}(\kappa,\lambda)|^2 \\ &= \int d\kappa d\lambda \lim_{t\to\infty} |\mathcal{M}_t(\kappa,\lambda) - \mathcal{M}(\kappa,\lambda)|^2 = 0 \end{aligned} \tag{6.18}$$

where we use by Lemma 6.1 dominated convergence to exchange the limit and the integral together with (6.17). It is known that $\lambda_{\max} = \|\mathcal{M}\| < 1$ for any fixed $\tilde{\sigma}$ (see, e.g., [48]). This, together with (6.18), implies that

$$\|\mathcal{M}_t\| \leq \|\mathcal{M}\| + \|\mathcal{M}_t - \mathcal{M}\| < 1 \tag{6.19}$$

for t large enough. □

Lemma 6.3. *Consider $\mathcal{Q}_t$ as defined in (6.5). There exists a $t_0 > 0$ such that uniformly for $t \geq t_0$ it holds*

$$|\mathcal{Q}_t(\kappa)| \leq c_3 e^{-\theta\kappa} \tag{6.20}$$

for any $\theta > 0$ and some constant $c_3 > 0$ (independent of t). Moreover,

$$\lim_{t\to\infty} \mathcal{Q}_t(\kappa) = \mathcal{Q}(\kappa) \tag{6.21}$$

uniformly for κ in a bounded set.

Proof. For the sake of this proof, set $\mathcal{J}_t := \mathcal{J}_t^{(0)}$ and $\mathcal{K}_t := \mathcal{K}_t^{(0)}$. First of all we prove that $\mathcal{Q}_t(\kappa)$ is uniformly bounded for $t \geq t_0$. Recall that $\mathcal{Q}_t(\kappa) = [(\mathbb{1} - \mathcal{M}_t)^{-1} \chi_{\tilde{\sigma}_t} J_{2t}](\kappa)$. Since $(\mathbb{1} - \mathcal{M}_t)^{-1}$ exists, we can use the identity

$$(\mathbb{1} - \mathcal{M}_t)^{-1} = \mathbb{1} + \chi_{\tilde{\sigma}_t} \mathcal{K}_t \chi_{\tilde{\sigma}_t} (\mathbb{1} - \mathcal{M}_t)^{-1}, \tag{6.22}$$

which upon integrating from $\tilde{\sigma}$ to ∞ against the function $\mathcal{J}_{2t}$ gives

$$\mathcal{Q}_t(\kappa) = \chi_{\tilde{\sigma}_t} \mathcal{J}_{2t}(\kappa) + \int_{\tilde{\sigma}_t}^{\infty} d\lambda \, \mathcal{K}_t(\kappa,\lambda) [(\mathbb{1} - \mathcal{M}_t)^{-1} \chi_{\tilde{\sigma}_t} \mathcal{J}_{2t}](\lambda). \tag{6.23}$$

Thus,

$$|\mathcal{Q}_t(\kappa)| \leq |\chi_{\tilde{\sigma}_t} \mathcal{J}_{2t}(\kappa)| + \int_{\tilde{\sigma}_t}^{\infty} d\lambda \, \big|\mathcal{K}_t(\kappa,\lambda)\big| \big|[(\mathbb{1} - \mathcal{M}_t)^{-1} \chi_{\tilde{\sigma}_t} \mathcal{J}_{2t}](\lambda)\big|. \tag{6.24}$$

But

$$|[(\mathbb{1}-\mathcal{M}_t)^{-1}\chi_{\tilde\sigma_t}\mathcal{J}_{2t}](\lambda)| \le \|(\mathbb{1}-\mathcal{M}_t)^{-1}\| \left|\mathcal{J}_{2t}\right|_\infty \tag{6.25}$$

is uniformly bounded for $t \ge t_0$ (by Lemma 6.1 and Lemma 6.2). Then, using the bound for $\mathcal{K}_t$ and $\mathcal{J}_t^{(s)}(\kappa)$ in (6.6) we obtain the bound (6.20).

To prove (6.21), we show that

$$|\mathcal{Q}_t-\mathcal{Q}|_\infty = \sup_\kappa |\mathcal{Q}_t(\kappa)-\mathcal{Q}(\kappa)| \to 0, \tag{6.26}$$

as $t\to\infty$. We have

$$\begin{aligned}|\mathcal{Q}_t-\mathcal{Q}|_\infty &= \left|(\mathbb{1}-\mathcal{M}_t)^{-1}\chi_{\tilde\sigma_t}\mathcal{J}_{2t}-(\mathbb{1}-\mathcal{M})^{-1}\chi_{\tilde\sigma}\mathrm{Ai}\right|_\infty\\ &\le \left|\left[(\mathbb{1}-\mathcal{M}_t)^{-1}-(\mathbb{1}-\mathcal{M})^{-1}\right]\chi_{\tilde\sigma}\mathcal{J}_{2t}\right|_\infty\\ &\quad+\left|(\mathbb{1}-\mathcal{M})^{-1}\left[\chi_{\tilde\sigma}\mathcal{J}_{2t}-\chi_{\tilde\sigma}\mathrm{Ai}\right]\right|_\infty+\mathcal{O}(t^{-1/3}),\end{aligned} \tag{6.27}$$

where the correction term $\mathcal{O}(t^{-1/3})$ comes from the fact that the difference between $\tilde\sigma_t$ and $\tilde\sigma$ is not larger than $(2t)^{-1/3}$. Then,

$$\begin{aligned}(6.27) &\le \left\|(\mathbb{1}-\mathcal{M}_t)^{-1}-(\mathbb{1}-\mathcal{M})^{-1}\right\| \,|\chi_{\tilde\sigma}\mathcal{J}_{2t}|_\infty\\ &\quad+\left\|(\mathbb{1}-\mathcal{M})^{-1}\right\|\,|\chi_{\tilde\sigma}\mathcal{J}_{2t}-\chi_{\tilde\sigma}\mathrm{Ai}|_\infty+\mathcal{O}(t^{-1/3})\end{aligned} \tag{6.28}$$

The first term goes to zero as $t\to\infty$. Indeed, $|\chi_{\tilde\sigma}\mathcal{J}_{2t}|_\infty \le C < \infty$ by Lemma 6.1, and, using the identity

$$(\mathbb{1}-\mathcal{M}_t)^{-1}-(\mathbb{1}-\mathcal{M})^{-1} = (\mathbb{1}-\mathcal{M}_t)^{-1}\left[\mathcal{M}_t-\mathcal{M}\right](\mathbb{1}-\mathcal{M})^{-1} \tag{6.29}$$

together with the fact that $\|\mathcal{M}_t\|<1$, $\|\mathcal{M}\|<1$, and $\|\mathcal{M}-\mathcal{M}_t\|\to 0$ in the $t\to\infty$ limit (see Lemma 6.2 and (6.18)); so one has $\|(\mathbb{1}-\mathcal{M}_t)^{-1}-(\mathbb{1}-\mathcal{M})^{-1}\|\to 0$. The second term goes to zero as well, since $\|(\mathbb{1}-\mathcal{M})^{-1}\|$ is bounded and, by Lemma 6.1, $|\chi_{\tilde\sigma}\mathcal{J}_{2t}-\chi_{\tilde\sigma}\mathrm{Ai}|_\infty\to 0$. ☐

Proof of Theorem 2.2, formula (2.20). We now define new functions $\mathcal{A}_t(s,\xi)$, $\mathcal{B}_t(s,\xi)$, $\mathcal{C}_t(s,\xi)$, which are rescaled versions of $A(\tau,x)$, $B(\tau,x)$, $C(\tau,x)$ (see formula (5.24)) under the scaling (6.1):

$$\begin{aligned}\mathcal{A}_t(s,\xi) &:= t^{1/3}A(st^{2/3},\xi t^{1/3}),\\ \mathcal{B}_t(s,\xi) &:= t^{1/3}B(st^{2/3},\xi t^{1/3}),\\ \mathcal{C}_t(s,\xi) &:= t^{1/3}C(st^{2/3},\xi t^{1/3}).\end{aligned} \tag{6.30}$$

As $t\to\infty$, these functions will converge to $\mathcal{A}(s,\xi),\mathcal{B}(s,\xi),\mathcal{C}(s,\xi)$ of (2.15) and (2.16).

One then recognizes in these expressions the functions (6.5), thus yielding

$$
\begin{aligned}
\mathcal{A}_t(s,\xi) &= \mathcal{J}_t^{(s)}(\sigma-\xi) \\
&\quad + \frac{1}{(2t)^{1/3}} \sum_{\kappa\in I_{n,t}} \frac{1}{(2t)^{1/3}} \sum_{\alpha\in(2t)^{-1/3}\mathbb{N}} \mathcal{Q}_t(\kappa)\mathcal{J}_{2t}^{(0)}(\kappa+\alpha)\mathcal{J}_t^{(s)}(2^{1/3}\alpha+\sigma-\xi), \\
\mathcal{B}_t(s,\xi) &= \frac{1}{(2t)^{1/3}} \sum_{\kappa\in I_{n,t}} \mathcal{Q}_t(\kappa)\mathcal{J}_t^{(s)}(\xi-\sigma+2^{1/3}\kappa-t^{-1/3}), \\
\mathcal{C}_t(s,\xi) &= \frac{2^{-1/3}}{(2t)^{1/3}} \sum_{\kappa\in I_{n,t}} \mathcal{Q}_t(\kappa)\left(\mathcal{J}_{2t}^{(2^{-2/3}s)}(\kappa-2^{-1/3}\xi)+\mathcal{J}_{2t}^{(2^{-2/3}s)}(\kappa+2^{-1/3}\xi)\right) \\
&\quad + \frac{2^{-1/3}}{(2t)^{2/3}} \sum_{\kappa,\lambda\in I_{n,t}} \mathcal{Q}_t(\kappa)\mathcal{Q}_t(\lambda)\left(\mathcal{K}_t^{(s)}(\kappa-2^{-1/3}\xi,\lambda)+\mathcal{K}_t^{(s)}(\kappa+2^{-1/3}\xi,\lambda)\right).
\end{aligned}
\tag{6.31}
$$

For instance, the function $J_{k+1+a}(4t)$ in $A(\tau,x)$ becomes, upon setting $a=\alpha(2t)^{1/3}$ and $\kappa:=(2t)^{-1/3}(k-4t)$,

$$
J_{k+1+a}(4t) = J_{[4t+(\kappa+\alpha)(2t)^{1/3}+1]}(4t) = (2t)^{-1/3}\mathcal{J}_{2t}^{(0)}(\kappa+\alpha). \tag{6.32}
$$

Notice that the sum over $k\geq n$ in the expressions (5.24) becomes a sum over $\kappa\in I_{n,t}$ with

$$
I_{n,t} := (2t)^{-1/3}(\{n,n+1,\ldots\}-4t). \tag{6.33}
$$

so that the condition $k\geq n=2m+1=4t+2\sigma t^{1/3}+1$ translates into $\kappa=(2t)^{-1/3}(k-4t)>2^{2/3}\sigma=\tilde{\sigma}$. Setting the summation variable $c=\gamma t^{1/3}$, rewrite the kernel (5.26) in Theorem 5.4, with the scaling (6.1)

$$
\begin{aligned}
&\frac{(-1)^{x_2}e^{4t_2}}{(-1)^{x_1}e^{4t_1}}\frac{H_{n+1}(0)}{H_n(0)}\widetilde{\mathbb{K}}_m^{\mathrm{ext}}(t_1,x_1;t_2,x_2) \\
&= -\mathbb{1}_{[s_1>s_2]}\frac{H_{n+1}(0)}{H_n(0)}t^{1/3}p_{(s_1-s_2)t^{2/3}}(\xi_1 t^{1/3},\xi_2 t^{1/3}) + \mathcal{C}_t(s_1-s_2,\xi_1-\xi_2) \\
&+ \frac{1}{t^{1/3}}\sum_{\gamma\in t^{-1/3}\mathbb{N}}\left(\begin{array}{l}
\mathcal{A}_t(s_1,\xi_1-\gamma)\mathcal{A}_t(-s_2,\xi_2-\gamma)+\mathcal{A}_t(s_1,-\xi_1-\gamma)\mathcal{A}_t(-s_2,-\xi_2-\gamma) \\
-\mathcal{A}_t(s_1,\xi_1-\gamma)\mathcal{B}_t(-s_2,\xi_2-\gamma)-\mathcal{A}_t(s_1,-\xi_1-\gamma)\mathcal{B}_t(-s_2,-\xi_2-\gamma) \\
-\mathcal{B}_t(s_1,\xi_1-\gamma)\mathcal{A}_t(-s_2,\xi_2-\gamma)-\mathcal{B}_t(s_1,-\xi_1-\gamma)\mathcal{A}_t(-s_2,-\xi_2-\gamma)
\end{array}\right) \\
&- \frac{1}{t^{1/3}}\sum_{\gamma\in t^{-1/3}\mathbb{Z}_-}\left(\mathcal{B}_t(s_1,\xi_1-\gamma)\mathcal{B}_t(-s_2,\xi_2-\gamma)+\mathcal{B}_t(s_1,-\xi_1-\gamma)\mathcal{B}_t(-s_2,-\xi_2-\gamma)\right)
\end{aligned}
\tag{6.34}
$$

In view of (2.10) we have $\lim_{t\to\infty}H_{n+1}(0)/H_n(0)=1$ and in the $t\to\infty$ limit, $(n-4t)/(2t)^{1/3}\to\tilde{\sigma}$. Notice that the sums with the preceding volume element, $1/t^{1/3}$ or $1/(2t)^{1/3}$ depending on the case, can be just thought of as

integrals with the integrand being piece-wise constant. What follows holds uniformly in t for $t \geq t_0$ where t_0 is a fixed constant. The exponential bounds of Lemma 6.1 and Lemma 6.3 imply that for any $\theta > 0$ there exists some $c > 0$ (the constant c depends on σ, which is however fixed)

$$|\mathcal{A}_t(s, -\xi)| \leq c\, e^{-\theta\xi} \quad \text{and} \quad \lim_{t\to\infty} \mathcal{A}_t(s, \xi) = \mathcal{A}(s, \xi). \tag{6.35}$$

Moreover $\mathcal{A}_t(s, \xi)$ tends to $\mathcal{A}(s, \xi)$ uniformly on bounded sets, by uniform convergence on bounded sets and dominated convergence of the integrand. Using the exponential bound of Lemma 6.3 and the fact that $\mathcal{J}_t$ is just bounded, we obtain similarly

$$|\mathcal{B}_t(s, \xi)| \leq c \min\{1, e^{-\theta\xi}\} \quad \text{and} \quad \lim_{t\to\infty} \mathcal{B}_t(s, \xi) = \mathcal{B}(s, \xi). \tag{6.36}$$

Finally, the exponential bounds of Lemma 6.1 and Lemma 6.3 imply that

$$|\mathcal{C}_t(s, \xi)| \leq c \quad \text{and} \quad \lim_{t\to\infty} \mathcal{C}_t(s, \xi) = \mathcal{C}(s, \xi), \tag{6.37}$$

where the last limit holds uniformly for ξ and s in bounded sets.

Using the bounds in (6.35), (6.36), and (6.37), one concludes that the integrands (summands) in (6.34) are uniformly bounded by functions which are integrable (summable). This is uniform for ξ, η and s in a bounded set. Then, by dominated convergence, we can take the limit inside, thus yielding (2.20). Finally, the Gaussian term in (2.20) comes from the known asymptotic (for $s > 0$),

$$\lim_{t\to\infty} t^{1/3} e^{-2st^{2/3}} I_{\xi t^{1/3}}(2st^{2/3}) = \frac{1}{\sqrt{4\pi s}} \exp\left(-\xi^2/(4s)\right), \tag{6.38}$$

which can be derived from a saddle point argument. □

7 Integral representation of the Tacnode kernel

To derive the double integral representation (2.21) of Theorem 2.2 there are two ways. One can use the Airy functions integral representations (A.7) together with

$$\int_0^\infty d\lambda\, e^{-\lambda(u-v)} = \frac{1}{u-v}, \quad \text{whenever } \Re(u-v) > 0. \tag{7.1}$$

This is quite straightforward but it requires several computations which are not reported here.

The second is to do a steep descent analysis starting from formula (5.25) in Lemma 2.1. Here we merely indicate a sketch of the saddle point argument (not a proof). The limits of the other terms have been discussed in the previous section. The main task here is to take the limit of this double integral, when $t \to \infty$, with the scaling

$$\begin{aligned} & n = 2m+1, \quad m = 2t + \sigma t^{1/3} \\ & z = -1 + \zeta t^{-1/3} \quad \text{and} \quad w = -1 + \omega t^{-1/3} \\ & x_i = \xi_i t^{1/3} \quad \text{and} \quad t_i = s_i t^{2/3}, \quad i = 1, 2. \end{aligned} \tag{7.2}$$

Also recall the definitions (2.17) of the Laplace transforms $\hat{\mathcal{Q}}(\zeta)$ and $\hat{\mathcal{P}}(\zeta)$, as well as the function $\mathcal{C}$ in (2.16). The reader is reminded of the steepest descent discussion in Section 4.1. For taking the limit of the extended kernel, we need the following Lemma.

Lemma 7.1. *Given the scaling (7.2) above, the following limits hold:*

$$\lim_{t\to\infty} e^{t(z-z^{-1})}(-z)^m = e^{\frac{\zeta^3}{3}-\sigma\zeta}, \tag{7.3}$$

and

$$\begin{aligned} & \lim_{t\to\infty} T_n(z^{-1}) = e^{-2\sigma\zeta}\hat{\mathcal{Q}}(\zeta), \quad \lim_{t\to\infty} T_n(w) = e^{2\sigma\omega}\hat{\mathcal{Q}}(-\omega) \\ & \lim_{t\to\infty} S_n(z^{-1}) = \hat{\mathcal{P}}(\zeta), \qquad\quad \lim_{t\to\infty} S_n(w) = \hat{\mathcal{P}}(-\omega), \end{aligned} \tag{7.4}$$

where $\hat{\mathcal{P}}$ and $\hat{\mathcal{Q}}$ are the Laplace transforms defined in (2.17). One also checks:

$$\lim_{t\to\infty} \frac{(-w)^{x_2-1}}{(-z)^{x_1}} = \frac{e^{\xi_1\zeta}}{e^{\xi_2\omega}} \quad \textit{and} \quad \lim_{t\to\infty} e^{-t_i(z+z^{-1}+2)} = e^{s_i\zeta^2}. \tag{7.5}$$

Proof. Letting $t \to \infty$; setting $n = 2m+1$, $m = 2t + \sigma t^{1/3}$, the critical point will be at $z, w = -1$ and thus the leading contribution will come from the neighborhood of the critical points, which suggests the scalings in z and w above. The Taylor expansion of the F-function (4.22) gives

$$\begin{aligned} e^{t(z-z^{-1})}(-z)^m &= e^{t(z-z^{-1})+m\log(-z)} = e^{tF(z)+\sigma t^{1/3}\log(-z)} \\ &= e^{tF(-1+\zeta t^{-1/3})+\sigma t^{1/3}\log(1-\zeta t^{-1/3})} = e^{\frac{\zeta^3}{3}-\sigma\zeta}(1+\mathcal{O}(t^{-1/3})). \end{aligned} \tag{7.6}$$

Setting in addition the scaling for t_i and x_i, one finds by Taylor expanding about $z = -1$ and $w = -1$ the limits (7.5). Introducing the running variable

$k = 4t + \kappa(2t)^{1/3}$, one gets

$$\begin{aligned}
\lim_{t\to\infty} T_n(z^{-1}) &= \lim_{t\to\infty} \sum_{k\geq n} \frac{Q_k}{(-z)^{k-n+1}} = \lim_{t\to\infty} \sum_{k\geq n} Q_k e^{-(k-n+1)\log(-z)} \\
&= \lim_{t\to\infty} (2t)^{-1/3} \sum_{\kappa\geq\tilde\sigma+(2t)^{-1/3}} (2t)^{1/3} Q_{4t+\kappa(2t)^{1/3}} e^{-(\kappa-\tilde\sigma)(2t)^{1/3}(-\zeta t^{-1/3})} \\
&= \int_{\kappa\geq\tilde\sigma} d\kappa\, \mathcal{Q}(\kappa) e^{(\kappa-\tilde\sigma)\zeta 2^{1/3}} = e^{-2\sigma\zeta}\, \hat{\mathcal{Q}}(\zeta)
\end{aligned} \tag{7.7}$$

and similarly

$$\lim_{t\to\infty} T_n(w) = e^{2\sigma\omega} \int_{\kappa\geq\tilde\sigma} d\kappa\, \mathcal{Q}(\kappa) e^{-\kappa\omega 2^{1/3}} = e^{2\sigma\omega}\, \hat{\mathcal{Q}}(-\omega). \tag{7.8}$$

The limit of the expression S_n, as in (4.10), involves $\bar h_k$, as in (4.24). Using the formula (4.24) for $\bar h_k(z^{-1})$ in terms of Bessel functions and Lemma 6.1, one checks, introducing the running variable $a = \mu(2t)^{1/3}$,

$$\begin{aligned}
\lim_{t\to\infty} \bar h_k(z^{-1}) &= -\lim_{t\to\infty} \sum_{a\geq 0} (-z)^a J_{k+a+1}(4t) \\
&= -\lim_{t\to\infty} (2t)^{-1/3} \sum_{\kappa\geq\tilde\sigma+(2t)^{-1/3}} e^{\mu(2t)^{1/3}\log(1-\zeta t^{-1/3})} \mathcal{J}_{2t}^{(0)}(\kappa+\mu) \\
&= -\int_0^\infty d\mu\, e^{-\mu\zeta 2^{1/3}} \mathrm{Ai}(\kappa+\mu).
\end{aligned} \tag{7.9}$$

Therefore, one finds

$$\begin{aligned}
\lim_{t\to\infty} S_n(z^{-1}) &= \lim_{t\to\infty} \langle Q, \chi_n \bar h(z^{-1})\rangle = \lim_{t\to\infty} \sum_{k\geq n} Q_k \bar h_k(z^{-1}) \\
&= \lim_{t\to\infty} (2t)^{-1/3} \sum_{\kappa\geq\tilde\sigma+(2t)^{-1/3}} (2t)^{1/3} \mathcal{Q}_t(\kappa) \bar h_k(z^{-1}) \\
&= -\int_{\kappa\geq\tilde\sigma} d\kappa\, \mathcal{Q}(\kappa) \int_0^\infty d\mu\, e^{-\mu\zeta 2^{1/3}} \mathrm{Ai}(\kappa+\mu) = \hat{\mathcal{P}}(\zeta).
\end{aligned} \tag{7.10}$$

This ends the proof of Lemma 7.1. □

Sketch of Proof of Theorem 2.2, formula (2.21). Since the sum in brackets in (5.25) is invariant under the involution $x_1 \leftrightarrow -x_2$ and $t_1 \leftrightarrow -t_2$, it suffices to consider the double integral, with the first term only. The second half

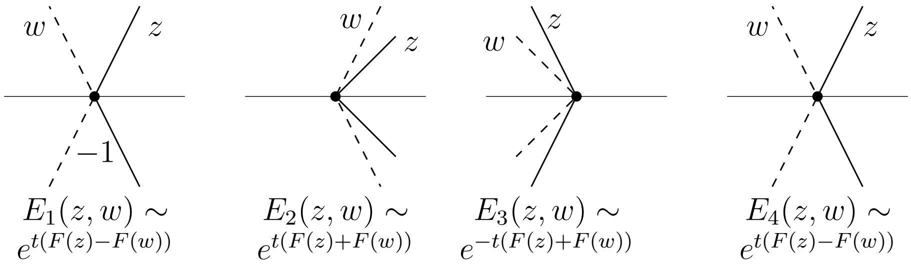

Figure 4: Contours $z \in \Gamma_0$ and $w \in \Gamma_{0,z}$ in the neighborhood of $z = w = -1$.

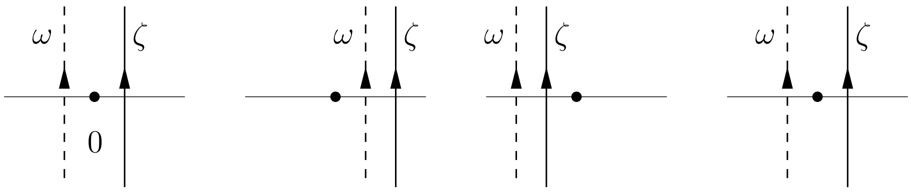

Figure 5: Vertical lines $\pm\delta + \mathrm{i}\mathbb{R}$ and $\pm 2\delta + \mathrm{i}\mathbb{R}$ of integration for ζ and ω.

comes for free by acting with the involution! Given scaling (7.2), Lemma 7.1 yields

$$\lim_{t\to\infty} t^{1/3} \frac{dz\ dw}{z-w} \frac{(-w)^{x_2-1}}{(-z)^{x_1}} \frac{e^{-t_1(z+z^{-1}+2)}}{e^{-t_2(w+w^{-1}+2)}} = \frac{d\zeta\ d\omega}{\zeta-\omega} \left(\frac{e^{\xi_1\zeta}}{e^{\xi_2\omega}}\right) \frac{e^{s_1\zeta^2}}{e^{s_2\omega^2}} \tag{7.11}$$

and from (4.12)

$$\begin{aligned} &\lim_{t\to\infty} \sum_{i=1}^{4} E_i(z,w) \\ &\qquad = \frac{e^{\frac{\zeta^3}{3}-\sigma\zeta}}{e^{\frac{\omega^3}{3}-\sigma\omega}} (1-\hat{\mathcal{P}}(\zeta))(1-\hat{\mathcal{P}}(-\omega)) - \frac{e^{\frac{\zeta^3}{3}-\sigma\zeta}}{e^{-\frac{\omega^3}{3}+\sigma\omega}} e^{2\sigma\omega}(1-\hat{\mathcal{P}}(\zeta))\hat{\mathcal{Q}}(-\omega) \\ &\qquad - \frac{e^{-\frac{\zeta^3}{3}+\sigma\zeta}}{e^{\frac{\omega^3}{3}-\sigma\omega}} e^{-2\sigma\zeta}(1-\hat{\mathcal{P}}(-\omega))\hat{\mathcal{Q}}(\zeta) - \frac{e^{\frac{\zeta^3}{3}-\sigma\zeta}}{e^{\frac{\omega^3}{3}-\sigma\omega}} \frac{e^{2\sigma\zeta}}{e^{2\sigma\omega}} \hat{\mathcal{Q}}(-\zeta)\hat{\mathcal{Q}}(\omega). \end{aligned} \tag{7.12}$$

Combining (7.11) and (7.12) yields the following limit below, first with the contours as indicated in Figure 4, which then can be transformed into the vertical lines above in Figure 5, compatible with Figure 4. Indeed, to pick steep descent paths about $z = w = -1$ respecting the integration contours in $\oint_{\Gamma_0} dz \oint_{\Gamma_{0,z}} dw$ of (7.13), one must choose the local paths, as illustrated in

Figure 4; these paths must be completed by closed contours encircling the origin deformed to provide steep descent contours. In the ζ,ω scale, there are 4 rays emanating from the origin $\omega=\zeta=0$; one is then free to deform these rays so as to obtain two parallel imaginary lines near the origin, as depicted above. Therefore the following limit holds for the first double integral

$$\begin{aligned}
&\lim_{t\to\infty}\frac{t^{1/3}}{(2\pi \mathrm{i})^2}\oint_{\Gamma_0}dz\oint_{\Gamma_{0,z}}\frac{dw}{z-w}\frac{(-w)^{x_2-1}}{(-z)^{x_1}}\frac{e^{-t_1(z+z^{-1}+2)}}{e^{-t_2(w+w^{-1}+2)}}\sum_{i=1}^{4}E_i(z,w)\\
=&\frac{1}{(2\pi \mathrm{i})^2}\int_{\delta+\mathrm{i}\mathbb{R}}d\zeta\int_{-\delta+\mathrm{i}\mathbb{R}}d\omega\,\frac{e^{\frac{\zeta^3}{3}-\sigma\zeta}}{e^{\frac{\omega^3}{3}-\sigma\omega}}\frac{e^{s_1\zeta^2}}{e^{s_2\omega^2}}\left(\frac{e^{\xi_1\zeta}}{e^{\xi_2\omega}}\right)\frac{(1-\hat{\mathcal{P}}(\zeta))(1-\hat{\mathcal{P}}(-\omega))}{\zeta-\omega} \quad (i)\\
&-\frac{1}{(2\pi \mathrm{i})^2}\int_{2\delta+\mathrm{i}\mathbb{R}}d\zeta\int_{\delta+\mathrm{i}\mathbb{R}}d\omega\,\frac{e^{\frac{\zeta^3}{3}-\sigma\zeta}}{e^{-\frac{\omega^3}{3}-\sigma\omega}}\frac{e^{s_1\zeta^2}}{e^{s_2\omega^2}}\left(\frac{e^{\xi_1\zeta}}{e^{\xi_2\omega}}\right)\frac{(1-\hat{\mathcal{P}}(\zeta))\hat{\mathcal{Q}}(-\omega)}{\zeta-\omega} \quad (ii)\\
&-\frac{1}{(2\pi \mathrm{i})^2}\int_{-\delta+\mathrm{i}\mathbb{R}}d\zeta\int_{-2\delta+\mathrm{i}\mathbb{R}}d\omega\,\frac{e^{-\frac{\zeta^3}{3}-\sigma\zeta}}{e^{\frac{\omega^3}{3}-\sigma\omega}}\frac{e^{s_1\zeta^2}}{e^{s_2\omega^2}}\left(\frac{e^{\xi_1\zeta}}{e^{\xi_2\omega}}\right)\frac{(1-\hat{\mathcal{P}}(-\omega))\hat{\mathcal{Q}}(\zeta)}{\zeta-\omega} \quad (iii)\\
&-\frac{1}{(2\pi \mathrm{i})^2}\int_{\delta+\mathrm{i}\mathbb{R}}d\zeta\int_{-\delta+\mathrm{i}\mathbb{R}}d\omega\,\frac{e^{\frac{\zeta^3}{3}+\sigma\zeta}}{e^{\frac{\omega^3}{3}+\sigma\omega}}\frac{e^{s_1\zeta^2}}{e^{s_2\omega^2}}\left(\frac{e^{\xi_1\zeta}}{e^{\xi_2\omega}}\right)\frac{\hat{\mathcal{Q}}(-\zeta)\hat{\mathcal{Q}}(\omega)}{\zeta-\omega}. \quad (iv)
\end{aligned} \tag{7.13}$$

In view of the scaling (7.2), the involution $x_1\leftrightarrow -x_2$ and $t_1\leftrightarrow -t_2$ induces the involution $\xi_1\leftrightarrow-\xi_2$ and $s_1\leftrightarrow -s_2$, so that the limit of the other double integral is given by the same formula (7.13) above, but with

$$\xi_1\leftrightarrow-\xi_2 \quad \text{and} \quad s_1\leftrightarrow-s_2. \tag{7.14}$$

We are also allowed to interchange the integration variables $\zeta\leftrightarrow-\omega$, provided the contours of integration are modified accordingly; this last interchange implies

$$\int_{\delta+\mathrm{i}\mathbb{R}}d\zeta\int_{-\delta+\mathrm{i}\mathbb{R}}d\omega \quad \text{remains} \tag{7.15}$$

$$\int_{2\delta+\mathrm{i}\mathbb{R}}d\zeta\int_{\delta+\mathrm{i}\mathbb{R}}d\omega \quad \text{and} \quad \int_{-\delta+\mathrm{i}\mathbb{R}}d\zeta\int_{-2\delta+\mathrm{i}\mathbb{R}}d\omega \quad \text{interchange.} \tag{7.16}$$

So, the three combined maps,

$$\zeta\leftrightarrow-\omega, \quad s_1\leftrightarrow-s_2, \quad \xi_1\leftrightarrow-\xi_2, \tag{7.17}$$

have the following effect on the four double integrals $(i),\dots,(iv)$ in (7.13):

$$\begin{aligned}
&\text{double integral (i) with } \tfrac{e^{\xi_1\zeta}}{e^{\xi_2\omega}} \to \text{same double integral (i), except for } \tfrac{e^{-\xi_1\zeta}}{e^{-\xi_2\omega}}\\
&\text{double integral (ii) with } \tfrac{e^{\xi_1\zeta}}{e^{\xi_2\omega}} \to \text{same double integral (iii), except for } \tfrac{e^{-\xi_1\zeta}}{e^{-\xi_2\omega}}\\
&\text{double integral (iii) with } \tfrac{e^{\xi_1\zeta}}{e^{\xi_2\omega}} \to \text{same double integral (ii), except for } \tfrac{e^{-\xi_1\zeta}}{e^{-\xi_2\omega}}\\
&\text{double integral (iv) with } \tfrac{e^{\xi_1\zeta}}{e^{\xi_2\omega}} \to \text{same double integral (iv), except for } \tfrac{e^{-\xi_1\zeta}}{e^{-\xi_2\omega}}.
\end{aligned}$$

Therefore the limit

$$
\begin{aligned}
\lim_{t\to\infty} \frac{t^{1/3}}{(2\pi \mathrm{i})^2} \oint_{\Gamma_0} dz \oint_{\Gamma_{0,z}} \frac{dw}{z-w} \sum_{i=1}^{4} E_i(z,w) \\
\times \left(\frac{(-w)^{x_2-1}}{(-z)^{x_1}} \frac{e^{-t_1(z+z^{-1}+2)}}{e^{-t_2(w+w^{-1}+2)}} + \frac{(-z)^{x_2}}{(-w)^{x_1+1}} \frac{e^{-t_1(w+w^{-1}+2)}}{e^{-t_2(z+z^{-1}+2)}} \right)
\end{aligned}
\tag{7.18}
$$

is given by r.h.s. of (7.13) with the replacement

$$
\frac{e^{\xi_1 \zeta}}{e^{\xi_2 \omega}} \longrightarrow \frac{e^{\xi_1 \zeta}}{e^{\xi_2 \omega}} + \frac{e^{-\xi_1 \zeta}}{e^{-\xi_2 \omega}}. \tag{7.19}
$$

Finally, in order to change the sign of the last integral, one switches the sign $\omega \to -\omega$ and $\zeta \to -\zeta$, which changes

$$
-\int_{\delta+\mathrm{i}\mathbb{R}} d\zeta \int_{-\delta+\mathrm{i}\mathbb{R}} d\omega \frac{1}{\zeta-\omega} \quad \text{into} \quad +\int_{-\delta+\mathrm{i}\mathbb{R}} d\zeta \int_{\delta+\mathrm{i}\mathbb{R}} d\omega \frac{1}{\zeta-\omega}. \tag{7.20}
$$

Renaming variables $\zeta \to u,\ \omega \to v$ gives formula (2.21).

A Some properties of Bessel and Airy functions

Let us recall that the Bessel function representation of order $n \in \mathbb{Z}$

$$
J_n(2t) = \frac{1}{2\pi \mathrm{i}} \oint_{\Gamma_0} dz \frac{e^{t(z-z^{-1})}}{z^{n+1}} \tag{A.1}
$$

has the symmetries

$$
J_n(2t) = (-1)^n J_{-n}(2t) = (-1)^n J_n(-2t). \tag{A.2}
$$

Moreover,

$$
\frac{1}{2\pi \mathrm{i}} \oint_{\Gamma_0} \frac{dz}{z} \frac{e^{b(z-z^{-1})} e^{a(z+z^{-1})}}{z^n} = \left(\frac{b+a}{b-a} \right)^{n/2} J_n\left(2\sqrt{b^2-a^2} \right). \tag{A.3}
$$

It is well-known [1] that

$$
\lim_{t\to\infty} t^{1/3} J_{[2t+\xi t^{1/3}]}(2t) = \mathrm{Ai}(\xi). \tag{A.4}
$$

An uniform bound obtained in [34] is

$$|(2t)^{1/3} J_n(2t)| \leq c, \quad c = 0.785\ldots, \quad n \in \mathbb{Z}. \tag{A.5}$$

This bound, together with uniform expansion which can be found in [1] is used in Lemma A.1 of [23] to get the following result. Fix any $\theta > 0$. Then, there exists a constant $t_0 > 0$ and a constant $C > 0$ such that, uniformly in $t \geq t_0$,

$$|t^{1/3} J_{[2t+\xi t^{1/3}]}(2t)| \leq C \min\{1, e^{-\theta\xi}\}. \tag{A.6}$$

Actually, the statement of Lemma A.1 of [23] is for $\theta = 1/2$ but inspecting the proof it is straightforward to see that it holds for any fixed $\theta > 0$. The Airy function has, between others, the following two integral representations. For any $\delta > 0$, it holds

$$\mathrm{Ai}(x) = \frac{1}{2\pi \mathrm{i}} \int_{\delta+\mathrm{i}\mathbb{R}} du\, e^{u^3/3-ux}, \quad \mathrm{Ai}(x) = \frac{1}{2\pi \mathrm{i}} \int_{-\delta+\mathrm{i}\mathbb{R}} dv\, e^{-v^3/3+vx}. \tag{A.7}$$

Moreover, for any $\delta > 0$, it holds

$$\begin{aligned} \mathrm{Ai}^{(s)}(x) &= e^{s\,x+2s^3/3}\mathrm{Ai}(x+s^2) = \frac{1}{2\pi \mathrm{i}} \int_{\delta+\mathrm{i}\mathbb{R}} du\, e^{u^3/3+u^2 s-ux}, \\ \mathrm{Ai}^{(s)}(x) &= e^{s\,x+2s^3/3}\mathrm{Ai}(x+s^2) = \frac{1}{2\pi \mathrm{i}} \int_{-\delta+\mathrm{i}\mathbb{R}} dv\, e^{-v^3/3+v^2 s+vx}. \end{aligned} \tag{A.8}$$

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